Recitation Notes: Week 3

Equations Involving Languages (and Regular Expressions)

One of the reasons we "re-use" some arithmetical symbols for strings and languages is that they obey similarly-stated laws. E.g. the <u>Distributive Law</u>: $a(b+c) = ab + ac$.

1. First note that when you have matrices A, B, C instead then this is really two laws, because matrix mult. does not commute:

$$A \cdot (B+C) = A \cdot B + A \cdot C$$
$$(B+C) \cdot A = B \cdot A + C \cdot A$$

which for matrices are not the same.

We get optically the same two laws for <u>languages</u> $A, B,$ and C:

$$A \cdot (B \cup C) = A \cdot B \cup A \cdot C$$
$$(B \cup C) \cdot A = B \cdot A \cup C \cdot A$$

To prove the latter one, a string x belongs to $(B \cup C) \cdot A$

$\iff$ x can be broken as $x =: y \cdot z$ with $y \in B \cup C$ and $z \in A$

$\iff$ x can be broken as $y \cdot z$ such that ($y \in B$ or $y \in C$) and $z \in A$.

$\iff$ x " " $y \cdot z$ such that ($y \in B$ and $z \in A$) or ($y \in C$ and $z \in A$)

$\iff$ x can be broken as $y \cdot z$ such that $y \in B$ and $z \in A$ or
x can be broken as $y' \cdot z'$ such that $y' \in C$ and $z' \in A$

$\iff$ $x \in B \cdot A$ or $x \in C \cdot A$ $\iff$ $x \in B \cdot A \cup C \cdot A$ $\boxtimes$

When α, β, and γ are regular expressions
alpha beta gamma
standing for languages, we can write the same laws as:

$$\alpha \cdot (\beta \cup \gamma) = \alpha \cdot \beta \cup \alpha \cdot \gamma$$
$$(\beta \cup \gamma) \cdot \alpha = \beta \cdot \alpha \cup \gamma \cdot \alpha.$$

For this reason, many people write + for union with regexps. (Don't confuse with superscript +.) This also helps when 0U1 might look too much like 001, and | for OR looks like 1

2. <u>Power Notation</u>: $A^2 = A \cdot A = \{x \cdot y : x \in A \wedge y \in A\}$
<u>Note</u>: This is <u>not</u> the same as $\{x \cdot x : x \in A\}$.
$A^3 = A \cdot A \cdot A$, $A^4 = A \cdot A \cdot A \cdot A$, etc. (Yes, $\cdot$ <u>is</u> associative)

<u>Law of Exponents</u>: For all $i, j \geq 0$, $\boxed{A^i \cdot A^j = A^{i+j}}$.

Now for a key point: We want this to hold even when $j = 0$
$A^i \cdot A^0 = A^{i+0} = A^i$.
In order for this to hold, we need $A^0 = \{\varepsilon\}$. We <u>define</u> this for all languages A, even $A = \emptyset$: $\emptyset^0 =_{def} \{\varepsilon\}$.

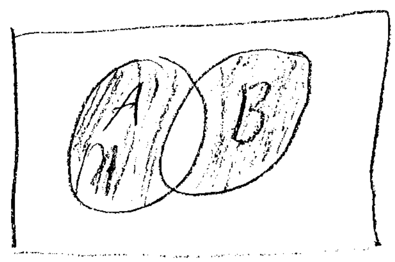
Note $A^* =_{def} A^0 \cup A^1 \cup A^2 \cup \cdots$ always has ε, since $A^0 = \{\varepsilon\}$
Thus $\emptyset^* = \emptyset^0 \cup \emptyset^1 \cup \emptyset^2 \cup \cdots = \{\varepsilon\} \cup \emptyset \cup \emptyset \cdots \underline{\{\varepsilon\}}$.

3. Some Things Are Different

In math, $a + a = 2a$, which is $> a$ when $a > 0$.
But with sets, $A \cup A = A$, no change!

We can write set difference $A - B$ like subtraction, as the text does. But we'll prefer $A \setminus B$ with <u>backslash</u>.

The symmetric difference is $(A \setminus B) \cup (B \setminus A)$.
It also equals $(A \cup B) \setminus (A \cap B)$.
Some people write $\underline{A \Delta B}$ for it.



Others write $\underline{A \oplus B}$ and use $\oplus$ for logical XOR too.
Note $A \Delta B = \{x : x \in A \ \underline{XOR} \ x \in B\}$.

<u>Challenge Q</u>: Does $A \cdot (B \oplus C) = (A \cdot B) \oplus (A \cdot C)$ always hold?

Tricky! $A = \{0, 00\}$, $B = \{1\}$, $C = \{01\}$ gives a <u>counterexample</u>:

$B \oplus C = B \cup C = \{1, 01\}$, so $A \cdot (B \oplus C) = \{01, 001, 0001\}$. But on the RHS:

$A \cdot B = \{01, 001\}$ The symmetric difference throws out 001 <u>entirely</u>
$A \cdot C = \{001, 0001\}$. $A \cdot B \ \Delta \ A \cdot C = \{01, 0001\} \neq A(B \Delta C)$

The failure of this law is one reason <u>not</u> to write $A \oplus B$, so $\underline{A \Delta B}$.

<u>4.</u> Some laws are just for languages, e.g. $\underline{A^0 \cdot A^0 = A^0}$, $A(BA)^* = (AB)^* A$.