CSE 396 | Extra to Sipser §2.1 (sor 2010) probably Mon. Mar 1 lecture | & Spring 2002 Fall 2001

Induction Proofs and Context-Free Grammars

Let G be a (context-free) grammar, and let E be a language defined "in prose," i.e. extensionally. We want to ascertain whether $L(G) = E$.

Def^n: • G is sound for E if $L(G) \subseteq E$, i.e. every terminal string derivable in the grammar belongs to the language.
• G is comprehensive* if $E \subseteq L(G)$, i.e. if G can generate every string in the language.

*The actual term in logic is "complete," but this would cause confusion with a later, different use of the term "complete,"
(namely, as in "NP-complete" in our order of topics.)

These terms arose in logic where G is a system of logic with axioms and deduction rules, $L(G)$ equals the theorems in this system, and E is the set of true statements in the system. Gödel's Incompleteness Theorem states that in this case, whenever G contains all the rules of elementary arithmetic (such as the commutative, associative, and distributive laws of $+$ and $\cdot$), $L(G)$ must be a proper subset of E, i.e. not all true statements are provable. The cases we will work on are ones where $L(G)$ does equal E, and we need to prove $L(G) \subseteq E$ and $E \subseteq L(G)$.

For soundness, i.e. $L(G) \subseteq E$, we want to induct on the grammar. There is a neat technique for doing so called Structural Induction (SI). For context-free grammars, the SI technique can be described simply:

I. Assign to each variable A in the grammar a property P_A of terminal strings x it may derive. You can phrase each such "meaning" P_A in the form "P_A: I derive strings x such that..."

II. For the start symbol, you want P_S to imply $x \in E$. Often you can take P_S to be "I derive strings x such that $x \in E$," but sometimes you need to make P_S a stronger property to make part III work.

III. For each production $A \to RHS$, not just those involving P_S, show that if every variable B in RHS derives a substring that satisfies P_B, then the resulting string satisfies P_A "on LHS."