CSE 396 Lecture Tue Apr 3 Spring 2018 ①

TopHat# 9130

# "Adversary Argument" Script for CFL Pumping Lemma

**Adv**: "I have a CFG $G$ such that $L(G) = L$." $L(G') = L(G) \setminus \{\epsilon\}$

**You**: "What is your '$n$'?" [It must be at most $2^{|V'|}$ where $G' = (V', \Sigma, R', S')$ is your $G$ in Chomsky NF.

**Adv**: "$n$".

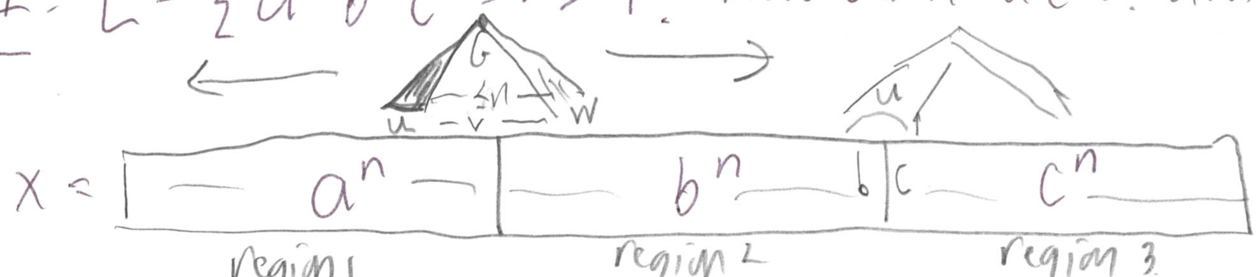
**You**: "We take $x =$ ________________." Note $x \in L$, and $|x| \geq n$

**You**: "Now give us a breakdown $x =: y \cdot u \cdot v \cdot w \cdot z$ s.t. $uw \neq \epsilon$ and $|uvw| \leq n$.

**Adv**: ________________ (you must be prepared for all possible cases of answers) (and show them).

**You**: "We take $i =$ ___. Then $x^{(i)} = yu^i v w^i z$ = ________________ is not in $L$ because ________________. This contradicts the (positive form of the) CFL Pumping Lemma. So you have no $G$. Thus $L$ is not a CFL."

**Example 1**: $L = \{a^k b^k c^k : k \geq 1\}$. Prove $L$ is not a CFL. Given "$n$", take




Then $x \in L$, but consider $x^{(0)}$ for all campaigns placement(s). Because $|uvw| \leq n$, $x^{(0)}$ cannot subtract chars from all three "regions". Because $uw \neq \epsilon$, $x^{(0)}$ subtracts from at least one region. Hence the three regions cannot stay equal in size, so $x^{(0)} \notin L$, so $L$ is not a CFL. ☒

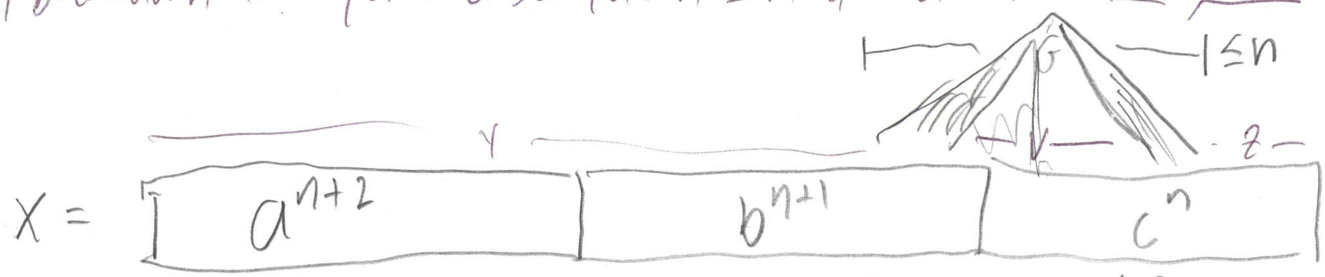
Word to the Wise on HW, the complement of $L$ (inside $a^*b^*c^*$) is a CFL! Hence unlike REG, the class $\mathcal{CFL}$ of CFLs is not closed under complements nor under $\cap$.

Example 2: $L_2 = \{a^i b^j c^k : i > j \wedge j > k\}$. Prove $L_2$ is not a CFL.

Proof: Given any $n > 0$, take $x = a^{n+2} b^{n+1} c^n$. Then $x \in L_2$.

Let any breakdown $x =: yuvwz$ s.t. $|uvw| \le n$ and $uw \ne \epsilon$ be given. Then

Cases:




In this case, $i = 0$ fails because it still leaves "$i > j > k$" within the defn of $L_2$, so $x^{(0)} \in L$

But $i = 2$ ("pumping up") succeeds because either $\#b(x^{(2)}) \geq \#a(x^{(2)})$ or $\#c \geq \#b$

To make this idea general and crisp, state the key observation:

Either (i) the compass does not overlap the region with $a^{n+2}$

or (ii) the compass does not overlap the region with $c^n$.

In case (i), we have $u \ne \epsilon$ or $w \ne \epsilon$, so $uw$ includes at least one $b$ or $c$

Hence pumping up to $i = 2$ adds at least one $b$ or $c$ keeping $a^{n+2}$ part the same

Can add: Subcase i.a: It adds a $b$. Then $\#a(x^{(2)}) > \#b(x^{(2)})$ is violated

Or $(x^{(2)})$ is not in $a^* b^* c^*$ at all.

i.b: It doesn't add a $b$ but does add a $c$. Then $\#b(x^{(2)}) \geq \#c(x^{(2)})$ is violated. Either way, $x^{(2)} \notin L_2$.

In case (ii), by $u \ne \epsilon$ or $w \ne \epsilon$, at least one $a$ or $b$ is subtracted with $c^n$ left alone

Subcase i.a: At least one $b$ is subtracted. This violates $\#b(x^{(0)}) > \#c(x^{0})$, i.e. "$j > k$"

i.b: $b$s stay the same but $a$'s go down: violates the "$i > j$" part.

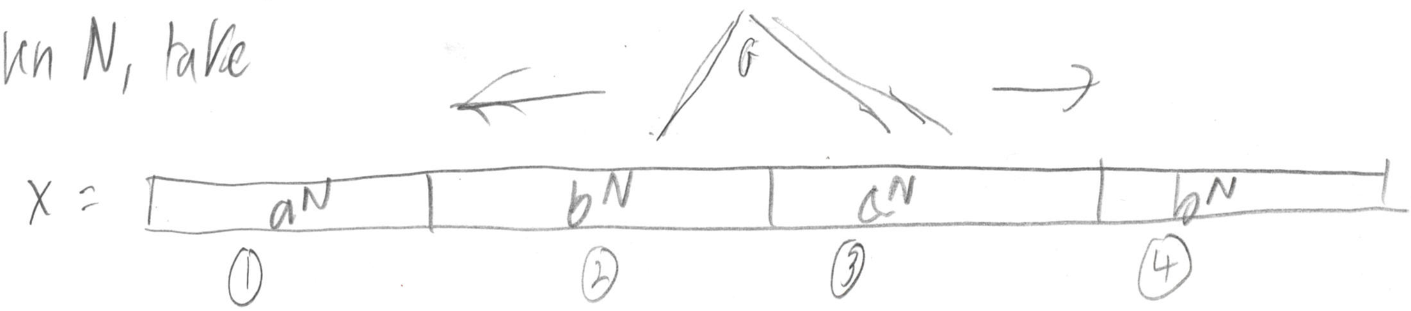
Either way, $x^{(0)} \notin L$. Since all these cases are collectively exhaustive, and in each case we get $x^{(0)} \notin L_2$ or $x^{(2)} \notin L_2$, this completes the script: $L_2$ is not a CFL.

Note, however, that $L_2$ too is the intersection of $\{a^i b^j : i > j\} \cdot c^*$ and $a^* \cdot \{b^j c^k : j > k\}$

Both of these are CFLs, so this is another case where CFLs are not closed under intersection

More Examples: $L_3 = \{a^m b^n a^m b^n : m,n \ge 0\}$. Prove not a CFL

Given N, take




in nonempty means

Observe: (A) The u,w arms of the compass must touch one of 1, 2, 3, or 4.
(B) By $|uvw| \le N$, the compass cannot touch both 1 and 3, nor both 2 and 4. Hence it cannot touch the other region as would be needed to "keep the balance" either between "$a^m$ and $a^m$" or "$b^n$ and $b^n$". Hence $x^{(0)} \notin L_3$, so $L_3 \notin$ CFL. ⊠

Example: $L_4 = \{a^i b^j a^k b^l : i > k \wedge j > l\}$. Not a CFL

Intuition: crossing dependencies   default: $i,j,k,l \ge 0$   Hardware: Combine ideas for $L_2$ & $L_3$.

But $L_5 = \{a^i b^j a^k b^l : i > k \vee j > l\}$ is a CFL, since it equals $\{a^i b^j a^k b^l : i > k\} \cup \{a^i b^j a^k b^l : j > l\}$.

$S_5 \to S_1 | S_2$   $S_1 \to S_3 B$   $S_3 \to a S_3 a | AB$   $B \to \varepsilon | bB$   $A \to a | a$

Hierarchy   CFG for second half from $S_2$ is similar, so $L_5$ is a CFL.

could be $S_1$

$L_6 = \{a^m b^m a^n b^n : m,n \ge 0\}$ is a CFL: $S_6 \to S_1 S_2$
$S_1 \to a S_1 b | \varepsilon$   $S_2 \to a S_2 b | \varepsilon$

$L_7 = \{a^i b^j a^k b^l : i > l \wedge j > k\}$

$S_7 \to a S_7 b | A S_8$   Operator is AND but dependencies are nested.

Is a CFL, designing a CFG for it is a self-study exercise: "Hierarchy"

END