"Adversary Argument" Script for (CFL Pumping Lemma)

**Adv:** "I have a CFG G such that \( L(G) = L \)."  \( L(G) = L(G) \cup \{ \varepsilon \} \)

**You:** "What is your 'n'?" (It must be at most 2.)  \( n \)

**Adv:** "\( n \)."

**You:** "We take \( X = \) \( \Rightarrow \) Note \( x \in L \), and \( |X| > n \)

**You:** "Now give us a breakdawn \( X = \varepsilon \)." \( \forall u,v,w,z \varepsilon \text{ s.t. } uvwz \leq n \).

**Adv:** (you must be prepared for all possible cases of answers) (and show them).

**You:** "We take \( i = \) \( \Rightarrow \) Then \( X^{(i)} = uv^i w^i z \) \( \Rightarrow \) is not in \( L \) because

This contradicts the (positive form of the) CFL Pumping Lemma. So you have no \( G \).

**You:** "Thus \( L \) is not a CFL.

**Example 1:** \( L = \{ \alpha^k \beta \gamma^k | k \geq 1 \} \) . Prove \( L \) is not a CFL. Given \( n \), take

\[ X = \varepsilon \]

Then \( x \in L \), but consider \( X^{(i)} \) for \( \forall i \), because \( |uvw| \leq n \).

\( \Rightarrow \) cannot subtract chars from all three "regions".

Because \( uvw \notin \varepsilon \), \( x^{(i)} \) subtracts from at least one region. Hence the three regions cannot stay equal in size, so \( X^{(i)} \notin L \), so \( L \) is not a CFL.

Word to the Wise: HW, the complement of \( L \) (inside \( \varepsilon \), \( ab \), \( c \)) is a CFL! Hence unlike \( \text{RED} \), the class \( \text{CFL} \) of CFLs is not closed under complements.
Example 2: \( L_2 = \{ a^i b^j c^k : i > j > k \} \). Prove \( L_2 \) is not a CFL.

Proof: Given any \( n > 0 \), take \( x = a^{n+2} b^n c^n \). Then \( x \in L_2 \).

Let any breakdown \( x = uvxyz \) st. \( | uvw| \leq n \) and \( uvw \neq \varepsilon \) be given. Then

Cases:

1. \( x = \frac{1}{2} a^{n+2} b^{n+1} c^{n} \)
2. \( x = \frac{1}{2} a^{n+2} b^{n} c^{n+1} \)

In this case, \( i = 0 \) fails because it still leaves \( \{ z \} \in L_2 \), so \( x^{(0)} \notin L_2 \)

But \( i = 2 \) ("pumping up") succeeds because either \( \# b(x^{(2)}) = \# a(x^{(1)}) \) or \( \# c(x^{(1)}) \) or

To make this idea general and crisp, state the key observation:

Either (i) the compass does not overlap the region with \( a^{n+2} \)

or (ii) the compass does not overlap the region with \( c^n \).

In case (i), \( \text{we have } u \neq \varepsilon \text{ or } w \neq \varepsilon, \text{ so } uvw \text{ includes at least one } b \text{ or } c \),

Hence pumping up to \( i = 2 \) adds at least one \( b \) or \( c \), keeping \( a^{n+2} \) part the same.

Subcase (1.a): \( \frac{1}{2} a^{n+2} b^{n+1} c^n \) adds at least one \( b \) or \( c \), keeping \( a^{n+2} \) part the same.

Subcase (1.b): \( \frac{1}{2} a^{n+2} b^n c^{n+1} \) does not add \( a \) but does add at least one \( b \).

In case (ii), \( u \neq \varepsilon \) or \( w \neq \varepsilon, \text{ at least one } a \text{ or } b \text{ is subtracted with } c^n \text{ left also} \).

Subcase (2.a): At least one \( b \) is subtracted. This violates \( \# b(x^{(1)}) > \# c(x^{(0)}) \), i.e., \( \{ i > j \} \).

Subcase (2.b): \( b \) bestays the same but \( a \)'s go down: violates the "\( i > j \)" part.

Either way, \( x^{(2)} \notin L_2 \). Since all these cases are collectively exhaustive, and in each case we get \( x^{(2)} \notin L_2 \), this completes the script: \( L_2 \) is not a CFL.

Note, however, that \( L_2 = \{ a^i b^j c^k : i > j > k \} \). Both of those are CFLs, so this is another case where CFLs are not closed under intersection.
Examples: \[ L_3 = \{ a^m b^n a^m b^n : m, n \geq 0 \} \text{ is not a CFL} \]

Given \( N \), take

\[ X = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
a^N & b^N & a^N & b^N
\end{array} \]

Observe:

A. The \( u, v \) arms of the compass must touch one of (1), (2), (3), or (4).
B. By \( |uw| \leq N \), the compass cannot touch both (1) and (3), nor both (2) and (4).
Hence it cannot touch the other region as would be needed to "keep the balance" either between "\( a^m \) and \( a^m \)" or "\( b^n \) and \( b^n \)." Hence \( X \not \in L_3 \), so \( L_3 \notin \text{CFL} \).

Example: \[ L_4 = \{ a^i b^j a^k b^l : i > k \} \]

A. Not a CFL

Intuition: Crossing dependencies

Hardware: Combine ideas for \( L_2 \) and \( L_3 \).

But \( L_5 = \{ a^i b^j a^k b^l : i > k, j > l \} \) is a CFL, since it equals \[ L_2 \cup \{ a^i b^j a^k b^l : i > k \} \text{ and } L_3 \].

\[ S_5 \rightarrow S_1S_2 \]
\[ S_1 \rightarrow S_3B \]
\[ S_3 \rightarrow aS_3a \mid AB \]
\[ B \rightarrow e \mid bb \]

A hierarchy.

\[ L_6 = \{ a^m b^n a^m b^n : m, n \geq 0 \} \text{ is a CFL: } S_6 \rightarrow S_1S_2 \]
\[ S_1 \rightarrow aS_3a \mid S_2 \]
\[ S_2 \rightarrow aS_3 \mid b \]

\[ L_7 = \{ a^i b^j a^k b^l : i > j \} \]

\[ S_7 \rightarrow aS_8 \mid bS_8 \]

Operator is AND but dependencies are nested.

Is a CFL, designing a CFG for it is a self-study exercise: "Hierarchy."