<u>Thm.</u> (Contrapositive)
Suppose for all $N > 0$,
if $\exists\, x \in L(G)$, $|x| > N$
s.t. for all breakdowns $x = yuvwz$ where $|uvw| \le N$ and $uw \ne \varepsilon$,
$\exists\, i \ge 0$ s.t. $x^{(i)} = yu^i v w^i z \notin L$,
then L is NOT a CFL.

Template:
Let any $N > 0$ be given,
Take $x =$ _____,
Consider any possible breakdown $x = yuvwz$ subject to $|uvw| \le N$ and $uw \ne \varepsilon$

Take $i =$ _____,
Then $x^{(i)} = yu^i v w^i z =$ _____ $\notin L$ because _____.
Since ~~such~~ N and the breakdown are arbitrary,
then L is not a CFL by CFL Pumping Lemma.

Example 1. $L = \{a^i b^j c^k : i<j<k\}$

Proof: Let any $N>0$ be given.

Take $x = a^N b^{N+1} c^{N+2}$.

Consider all the possible breakdowns s.t. $x = yuvwz$ with $|uvw| \le N$ and $uw \neq \varepsilon$

(1) $uvw = a^n$ (b^n, c^n), $n \le N$,

(2) uvw has r a's and s b's collectively

(3) uvw has at lease one c's

(1) * "pumping up" to (worst case $u=a$, $vw=\varepsilon$)

$$x^{(N+1)} = yu^i v w^i z = y \cdot a^{N+1} b^{N+1} \cdot c^{N+2} \notin L.$$

* for $uvw = b^n$,

"pumping down" to $x^{(0)} = yvz$.

since uvw has no a's or c's,

then $x^{(0)} = yvz = a^N \cdot b^{N+1-n} \cdot c^{N+2} \notin L(G)$ by $uw \neq \varepsilon$.

$\Downarrow$ at least reduce 1 b's.

* for $uvw = c^n$,

similarly, "pumping down" to $x^{(0)} = yvz$.

(2) Either r, s could be 0, not both.

Hence, for $i \ge 2$, $x^{(i)} = yu^i v w^i z$ has $N + (i-1)\cdot r$ a's and $N+1+(i-1)\cdot s$ b's, still $N+2$ c's

since $r, s \ge 1$,

Take $i=3$, $x^{(3)}$ has at least $N+2$ a's and $N+3$ b's. $\Rightarrow x^{(3)} \notin L$

(3) "pumping down" to $x^{(0)} = yvz$.

then $x^{(0)}$ has less than $N+2$-many c's.

$\Rightarrow x^{(0)} \notin L$.

Overall, all possible cases violate $(\forall i)\ x^{(i)} \in L$

$\Downarrow$

$\exists i,\ x^{(i)} \notin L$.

Therefore, L is not CFL.

Anothe way to consider all the breakdowns:

(1) $u = a^m$ for some $m > 0$,

then $x^{(2)} = yu^2vw^2z \notin L$, since $\#a(x^{(2)})$ is no less than b's.

(2) $u = b^m$ forsome $m > 0$,

then $x^{(0)} \notin L$

since # of a's is not less than # of b's.

(3) $u = c^m$, $m > 0$,

then $x^{(0)} \notin L$ since # of b's is not less than # of c's.

(4) $u = \varepsilon$ and replace u by w in the above three cases.

Example 2

$L = \{a^m b^n a^m b^n\}$

Proof: Let any $N > 0$ be given,

take $x = \underline{a}^N \underline{b}^N \underline{a}^N \underline{b}^N = yuvwz$ with $|uvw| \le N$

and $uw \neq \varepsilon$

idea: $\underline{uvw}$ must touch at least ~~one~~ of the four intervals,

and at most two

so for all possible cases:

"pumping down" to $x^{(0)} = yvz$ will reduce at least ~~one~~

one of the a's of b's,

then $x^{(0)} \notin L$.

$\Longrightarrow$ ~~overall,~~ Overall, L is not a CFL.

Q: What kind of model can recognize all those languages?

(regular, CFL, not CFL).

① Allow to change a char

② Allow to ~~mov~~ move left.

Allowing only ① or ② doesn't help.

Allowing both define a Turing Machine.

A Turing Machine (TM) or Deterministic TM (DTM),
(there is ~~alas~~ also an nondet. one, NTM)
liberalizes a DFA by
* allowing to change chars ~~one~~ one or more tapes.
* allowing tape head to move left (L) or stay (S) besides moving right (R).

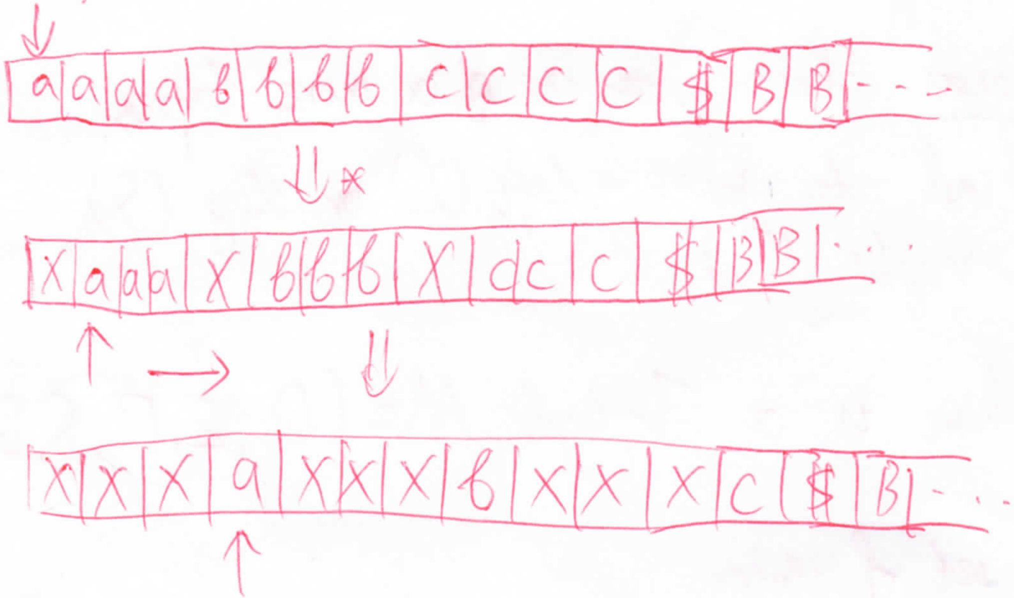
Def. A Turing Machine is a 7-tuple $M=(Q, \Sigma, \Gamma, \delta, B, s, F)$ where:
* Q is a finite set of states
* Σ is the finite input alphabet
* Γ is the ~~tape alph~~ work alphabet, where $B \in \Gamma$ and $\Sigma \subseteq \Gamma$.
* B is the blank (␣ in text, or 10, " " etc.)
* s is the start ~~state~~ state (q_0 in text)
* F is the set of ~~dis~~ desired final states
  (in Text, $F=\{q_{acc}\}$ where also W.L.O.G. there is a unique q_{rej} (rejecting state))
* $\delta \subseteq Q \times \Gamma \overset{'/'}{\times} \Gamma \times \{L, R, S\} \times Q$ (in text, only $\{L, R\}$)

Diagram: $p \xrightarrow{(c/d,\, D)} q$

typical tuple or instruction $(\underset{p \in Q}{p}, \underset{c \in \Gamma}{c} \underset{('/')}{,} \underset{d \in \Gamma}{d}, \underset{\text{direction}}{D}, q)$

TM can decide languages like $\{a^n b^n c^n : n \geq 1\}$. (not CFL)

Idea:



Furthermore:

* M is <u>deterministic</u> if for all $p \in Q$ and $c \in \Gamma$, there is <u>at most one</u> tuple (instruction) in δ that begins $(p, c/\ldots)$.
* M is "completed" if for all $p \neq q_{acc}, q_{rej}$ and $c \in \Gamma$, there is a tuple beginning $(p, c/\ldots)$

Together $\Rightarrow \delta$ is a function from $(Q \setminus \underbrace{\{q_{acc}, q_{rej}\}}_{\text{Halting states}} \times \Gamma)$

to $(\Gamma \times \{L, R, S\} \times Q)$

Otherwise, if $\exists$ any pair (q, c) with two or more tuples beginning $(q, c/\ldots)$, then M is an <u>NTM</u>.