Three Messages:

1. Undecidability is "Infectious."
2. --but not as much a barrier as we thought 40 and 80 years ago (Turing 1936)
3. Reductions include positive aspects of "negative" results.

Example: **$HALT_{TM}$**:

INST: A TM M (default = deterministic), an input x to M — same TYPE of instance as for A_{TM}

QUES: Does $M(x)\downarrow$? ("halt") — Technically different question from A_{TM} (historically identified)

Theorem The $HALT_{TM}$ language is undecidable.

The language $HALT_{TM} = \{\langle M, x\rangle : M(x)\downarrow\}$.

Proof: Suppose we had a total TM Q s.t. $L(Q) = HALT_{TM}$. Then we could use Q to get a total TM R s.t. $L(R) = A_{TM}$ as follows:

R: ↓ $\langle M, x\rangle$

Later, this will be done by a reduction

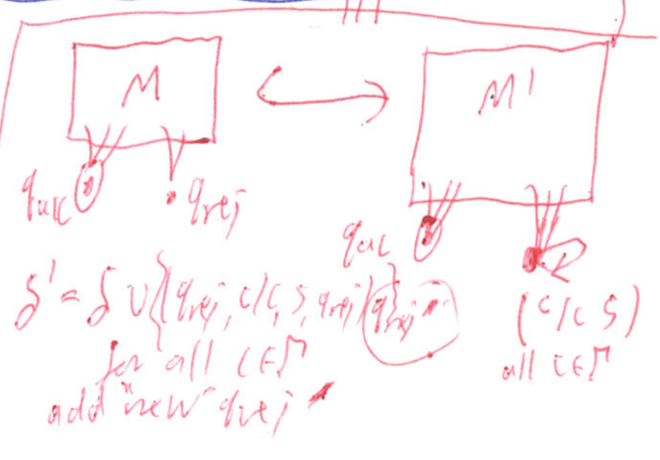
[Convert M to M' s.t. if and when M goes to q_{rej}, M' loops instead]

// The effect of this is that for all x,
// $M'(x)\downarrow \iff M$ accepts x

↓

[Feed $\langle M', x\rangle$ to Q] (By assumption, a ~~square~~ solid box.)

If Q accepts $\langle M', x\rangle$ accept, else reject.



Then R accepts $\langle M, x\rangle \iff Q$ accepts $\langle M', x\rangle$ — no longer true for real R' ②

$\iff M'(x)$ ~~accepts~~ halts $\iff M(x)$ accepts $\iff \langle M, x\rangle \in A_{TM}$.

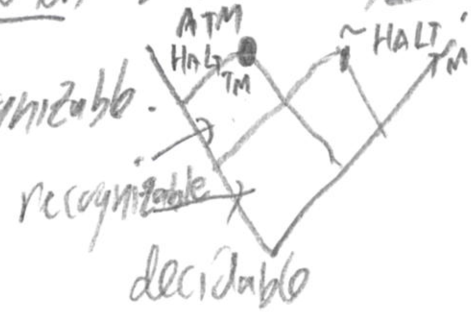
$\therefore R$ is total and $L(R) = A_{TM}$, <u>but this is impossible</u>.

(Last lecture showed A_{TM} is undecidable)

Hence there is no such Q. ☒

<u>Note</u>: $HALT_{TM}$ <u>is</u> recognizable — We <u>can</u> code R' to <u>run</u> $M(x)$ and accept <u>if</u> it halts. But if $M(x)\uparrow$, then our R' won't halt either. <u>Contradiction</u> is Q & R being <u>total</u>.

$\therefore$ The complement $\widetilde{HALT}_{TM}$ is not even recognizable.



<u>Example 2</u>: <u>Emptiness</u> <u>and</u> <u>Nonemptiness</u>

$NE_{TM} \doteq$ INSTANCE: A Turing Machine M ⟵ "Just an M" (TYPE)
QUESTION: Is $L(M) \neq \emptyset$?

tear

E_{TM}: INST: An M.
QUES: Is $L(M) = \emptyset$?

As languages: $NE_{TM} = \{\langle M\rangle : L(M) \neq \emptyset\}$
$E_{TM} = \{\langle M\rangle : L(M) = \emptyset\}$.

If "$\langle M\rangle$" includes all strings, then E_{TM} literally $= \sim NE_{TM}$.

If we consider any invalid code to yield the empty language, then $E'_{TM} = \{x : x \text{ is invalid or } x = \langle M\rangle \text{ s.t. } L(M) = \emptyset\} = \widetilde{NE}_{TM}$

Generally it will be OK to ignore the issue of invalid codes.

<u>Theorem</u>: NE_{TM} is recognizable but undecidable and so E_{TM} is not even recognizable.

Proof Suppose we had a total Q st. $L(Q) = NE_{TM}$.

Then we could build a total TM R deciding A_{TM} as follows:

R:

↓ input to R is $\langle M, x \rangle$

[Build M' that takes M and x as a fixed subroutine : "On any input w, call $M(x)$ if & when $M(x)$ accepts, accept, (else don't)"]

Computability of the conversion ✓

↓ M' - instance machine of the kind the NE_{TM} problem is about

[Q]

if Q acc. accept — else reject

Construction

M, x ⟶ M': ↓ input w {Ignore w for now. Write down x and simulate M on it.} — if and when $M(x)$ accepts ⟶ Accept.

Always bear in mind $M(x)$ might not halt.

The first part needs to build M' so that

Analysis: Correctness M accepts $x \Leftrightarrow L(M') \neq \emptyset$

$\langle M, x \rangle \in A_{TM} \Rightarrow M(x)$ accepts $\Rightarrow$ for all $w \in \Sigma^*$ $M'(w)$ eventually gets to accept $\Rightarrow L(M') = \Sigma^* \Rightarrow L(M') \neq \emptyset$

$\langle M, x \rangle \notin A_{TM} \equiv M(x)$ does not accept $\Rightarrow$ for all $w \in \Sigma^*$, $M'(w)$ never gets to accept $\Rightarrow L(M') = \emptyset$.

$\therefore L(R) = A_{TM} \wedge R$ is total, contradiction. So a total Q st. $L(Q) = NE_{TM}$ does not exist. ∎

Consequences: Not only is E_{TM} also undecidable — indeed not even r.e. but also: • ALL_{TM} ← Thursday

ALL_{TM}: Instance A TM M
QUESTION: Is $L(M) = \Sigma^*$?

Theorem: ALL_{TM} is undecidable.

Proof Suppose we had a total TM Q' st. $L(Q') = ALL_{TM}$. Using Q' in place of Q makes R behave the same. ie. $L(R) = A_{TM}$, contradiction, so Q' does not exist. ∎

Note: this does not say that ALL_{TM} is or isn't r.e. (c.e.)

Compare: $E_{CFG} = \{\langle G \rangle : L(G) = \emptyset\}$ (CFGs) is decidable

But we will see $ALL_{CFG} = \{\langle G \rangle : L(G) = \Sigma^*\}$ is undecidable.