CSE 396 — Lecture Tue 4/30 — Spring 2019

TopHat # 3334

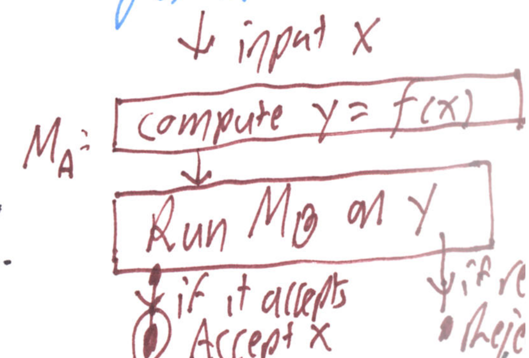
Defn: Given two languages $A, B \subseteq \Sigma^*$, say that A {mapping- / many-one} reduces to B, written $A \le_m B$, if there is a total computable function $f : \Sigma^* \to \Sigma^*$ such that for all $x \in \Sigma^*$, $x \in A \iff f(x) \in B$. (*)

Proposition: $A \le_m B$ if and only if $\tilde{A} \le_m \tilde{B}$.
because $x \notin A \iff f(x) \notin B$ if (*) holds.

Proposition $A \le_m A$ via f = the identity fn, so the $\le_m$ relation is reflexive.

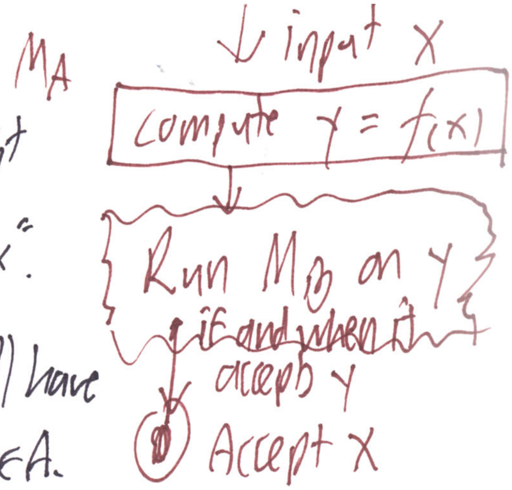
Proposition: If $A \le_m B$ via f and $B \le_m C$ via a function g, then $A \le_m C$ via the function $g \circ f$. Because $x \in A \iff f(x) \in B \iff g(f(x)) \in C$.

Lemma: Suppose $A \le_m B$. Then:
(a) If B is decidable, then so is A.
(b) If B is recognizable, then so is A.
(c) If B is co-c.e. then so is A.



Proof: (a) By B being decidable, we can take a total TM M_B s.t. $L(M_B) = B$. **Goal:** build a total TM M_A such that $L(M_A) = A$.
Then M_A is total and M_A accepts $x \iff M_B$ accepts y $(= f(x))$
$\iff f(x) \in B$ by $L(M_B) = B$
$\iff x \in A$ by $A \le_m B$.
So M_A accept $x \iff x \in A$ and M_A is total, so A is decidable.

ⓑ Suppose B is merely c.e. Then we can take M_B s.t. $L(M_B) = B$, but M_B might not be total. Can diagram using "fuzzy box".

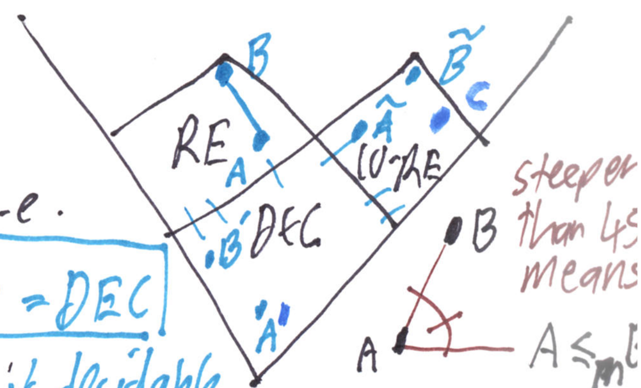


Then M_A might not be total either, but we still have M_A accepts $x \Leftrightarrow M_B$ accepts $y \Leftrightarrow y \in B \Leftrightarrow x \in A$.
So $L(M_A) = A$, so A is recognizable/c.e./r.e./etc.

ⓒ Suppose B is co-c.e. This means $\tilde{B}$ is c.e. By $A \leq_m B$, we also have $\tilde{A} \leq_m \tilde{B}$. By part (b), $\tilde{A}$ is c.e. Thus A is co-c.e. ☒

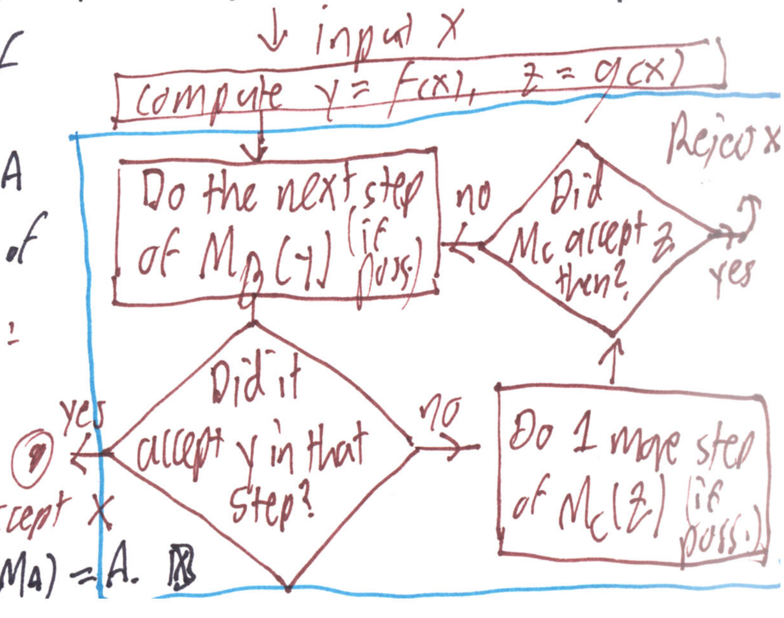
<u>Theorem</u>: If $A \leq_m B$ where B is c.e. and $A \leq_m C$ where C is co-c.e. then A is decidable.

RE ∩ coRE = DEC



<u>Consequence</u>: If A is c.e. and co-c.e, then A is decidable.
Proof of this consequence: Take $B = A$, $C = A$ and use identity reductions.

Proof: Take TMs M_B s.t. $L(M_B) = B$ and M_C s.t. $L(M_C) = \tilde{C}$, and take functions f, g s.t. $A \leq_m B$ via f and $A \leq_m C$ via g. Build a TM M_A that does step-by-step simulations of M_B and M_C in tandem. Then M_A:



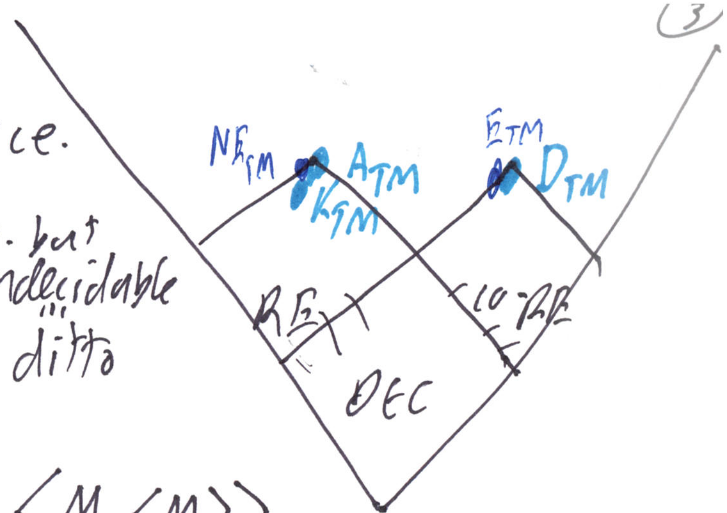
M_A <u>is total</u> because $x \in A \Rightarrow y \in B \Rightarrow M_B$ will eventually accept y, while $x \in \tilde{A} \Rightarrow z \in \tilde{C} \Rightarrow M_C$ will eventually accept z. And $L(M_A) = A$. ☒

Examples of Reductions

$D_{TM} = \{\langle M\rangle : \langle M\rangle \notin L(M)\}$ not c.e.

$K_{TM} = \{\langle M\rangle : \langle M\rangle \in L(M)\}$ c.e. but undecidable

$A_{TM} = \{\langle M, w\rangle : M$ accepts $w\}$. ditto



- $K_{TM} \leq_m A_{TM}$ via $f(\langle M\rangle) = \langle M, \langle M\rangle\rangle$.
- Here f is computable because it just duplicates whatever code it's given
- and f is correct because $\langle M\rangle \in K_{TM} \equiv M$ accepts $\langle M\rangle \longleftrightarrow \langle M, \langle M\rangle\rangle \in A_{TM}$.

Per the diagram, D_{TM} does NOT $\leq_m$ to K_{TM}, which is good because K_{TM} is c.e. so D_{TM} would be c.e., which it's not.

- NE_{TM}: INST: A TM M QUES: Is $L(M) \neq \emptyset$? As a language, $NE_{TM} = \{\langle M\rangle : L(M) \neq \emptyset\}$.

$A_{TM} \leq_m NE_{TM}$: We need to build a computable mapping of code

$\langle M, w\rangle \xrightarrow{f} M'$

such that $L(M') \neq \emptyset \Leftrightarrow M$ accepts w.

M':
0. input x (ignored for now)
1. Run M on w
2. if and when it accepts
3. Accept x.

$\therefore E_{TM}$ is also undecidable. NE_{TM} is c.e., so E_{TM} is co-c.e.

Study: $A_{TM} \leq_m K_{TM}$

f is computable because we can just drop a particular M and w into the slot in our code framework for M'.

f is correct because $\langle M, w\rangle \in A_{TM} \Rightarrow M$ accepts $w \Rightarrow$ for all x, M' accepts x $\Rightarrow L(M') = \Sigma^* \Rightarrow L(M') \neq \emptyset \Rightarrow \langle M'\rangle \in NE_{TM}$

whereas $\langle M, w\rangle \notin A_{TM} \Rightarrow$ for all x, $M'(x)$ does not escape the fuzzy box $\Rightarrow L(M') = \emptyset \Rightarrow f(M, w) = \langle M'\rangle \notin NE_{TM}$.

$\therefore A_{TM} \leq_m NE_{TM}$, and since A_{TM} is undecidable, so is NE_{TM}.

$\therefore A_{TM} \leq_m ALL_{TM} \equiv \{\langle M\rangle : L(M) = \Sigma^*\}$. So ALL_{TM} is undecidable too. $\Rightarrow \langle M'\rangle \notin ALL_{TM}$