Top Hat: \( \text{ALL}_{TM} = \{ \langle M \rangle : L(M) = \Sigma^* \} \) and \( \text{TOT} = \{ \langle M \rangle : M \text{ is total} \}

Constructions:

- \( M_1 \) (cf.) \( M_2 \)

- \( M_3 \) (cf.) \( M_4 \)

- \( M_3 \) on \( x \)
  - \( \text{loop} \ (c/1, s) \)
  - \( \text{fail if rej} \)

- \( \text{fall} \ (c/1, s) \)

- \( \text{old rej} \)

Analysis: On any \( x \), don't halt.

\( M_1(x) \) accepts \( \Rightarrow M_2(x) \) halts and accepts
\( M_1(x) \) rejects or does not halt \( \Rightarrow M_2(x) \uparrow \)
\( \therefore M_2 \) is total \( \Rightarrow L(M_1) = \Sigma^* \)
\( \therefore \text{ALL}_{TM} \leq_m \text{TOT} \)

\( M_3(x) \) halts \( \Rightarrow M_4(x) \) accepts \( x \) in either of two ways
\( M_3(x) \uparrow \Rightarrow M_4(x) \) doesn't halt either.
\( L(M_4) = \Sigma^* \Rightarrow M_3 \) is total.
\( \therefore \text{TOT} \leq_m \text{ALL}_{TM} \)

\( \therefore \text{ALL}_{TM} \equiv_m \text{TOT} \)

Intuition: Both require
\((\forall x \in \Sigma^*)(\exists \text{ a computation } \overline{c}) \overline{c} \text{ is valid and accepts/halts.} \)
Turing Machine Computations As Strings

Enough to do for 1-tape TMs. (nondeterministic OK)

An ID can be represented as \( I = [uqv] \)
over the alphabet \( Q \cup \Gamma \cup \{C, \#\} \).

Recall the relation \( I \downarrow M \).

We can write computations as

\[ I_0 \] \[ I_1 \] \[ I_2 \] \[ I_3 \] \cdots \[ I_t \]

or "ripple style" as

\[ I_0 \] \[ I_1 \] \[ I_2 \] \[ I_3 \] \[ I_4 \] \cdots \[ I_t \]

Define (either can be called ACH or "Accepting Computation"

\( V_M = \{ \varepsilon \text{ written normally: } \varepsilon \text{ is a valid finite } \}
\)

\( W_M = \{ \varepsilon \text{ written ripple style: accepting computation } \}
\)

Facts: (Much proved in th text)

- \( V_M \) and \( W_M \) are both complements of CFLs.
- \( W_M \) is the \( \cap \) of two CFLs.
- \( V_M \) can be recognized by a DFA with 2 Heads.
- One checks \( f_{ij} \cdot f_{hi} \) for each other for \( j \) odd. Line marked false
- \( \Rightarrow V_M \) can be recognized by a Linear Bounded Automaton.
Analysis: \( L(M) = \emptyset \iff V_m \) and \( W_m \) are empty.

And whenever \( L(M) \neq \emptyset \), \( V_m \) is not a CFL, \( W_m \) ditto.

Theorem: \( \text{E}_{TM} \leq_m \text{E}_{TM} \) reduces to all of these problems:

- All CFG
- All NPDAs
- All OPDAs

Inst: \( \theta_{OPDA}, p_1 \), and \( p_2 \)

\( L(\theta_{OPDA}) \cup L(p_1) \cup L(p_2) = \emptyset \)

EQ

CFG is likewise undecidable, not co-either.

Reduce All CFG \( G \rightarrow (G, G') \) where \( L(G) = \Sigma^* \)

\( S \rightarrow aSbS1 \epsilon \)

(similar, \( \text{ALL}_{TM} \leq_m \text{EQ}_{TM} \) so \( \text{EQ}_{TM} \) is neither co- nor co-either.

\( \text{ETM} \leq_m \text{EDLBA} \)

Both DLBA and NLGA are closed \( \leq_m \) ALL DLBA

\( \text{ENLGA} \)

under \( \text{E} \), so \( \text{ALL}_{NLGA} \)

\( \text{NE}_{TM} \leq_m \text{NE}_{DLBA} \leq_m \text{NE}_{2\text{Head DPA}} \)

How about All OPDA? Decidable since OPDAs can be complemented.

How about All NFA? Decidable by NFA-1 DFA

but how hard is it? \( \implies \text{Completeness Theory} \).
Defn: A TM $M$ runs in time $t(n)$ if for all $x \in \Sigma^*$, taking $n = |x|$, there exists a valid computation $I_0(x), I_1, I_2, \ldots, I_t$ with $t \leq t(n)$ within.

If $M$ is nondet, can demand all valid computations have $\leq t(n)$ steps before halting.

Defn: $P = \{L \mid \text{L is recognized by OTMs that run within time } p(n) \text{ for some polynomial } p \}$.

$NP = \{L \mid \text{by NTMs that run in } p(n) \text{ time} \}$

$A \leq_P m B$ if there is a polynomial-time computable function $f: \Sigma^* \rightarrow \Sigma^*$ such that $A \leq f(A) \in B$.

$CFL \leq_P \text{ by CYK alg.}$

Thursday will show completeness for NP under $\leq_P$ reduction.