Name and St.ID#:____

CSE439 Prelim II Fall 2024

Open book, open notes, closed neighbors, 75 minutes. The exam totals 100 pts., subdivided as shown. Do *all four problems* on these exam sheets—there is no "choice" option. em Show your work—this may help for partial credit. Use of a quantum circuit simulation app—whether online or downloaded—or matrix calculator is forbidden. You may use—and cite—theorems and facts from lectures and homeworks without further justification.

Notation: You may freely mix "standard linear algebra notation" and "Dirac *bra-ket* notation" for quantum states. For example, e_{10} means the same thing as $|10\rangle$ and is represented (in big-endian notation) by the unit column vector $[0, 0, 1, 0]^T$. Non-unit vectors should only use the standard notation. For matrix outer-products the *ket-bra* form typified by $|+\rangle\langle-|$ **is** standard. The notation $\langle u \mid v\rangle$ is the same as $\langle u, v\rangle$ for inner product. Quantum state vectors use big-endian notation by default; if you switch to little-endian you must say so.

(1) ($5 \times 5 = 25$ pts.) *True/False* **with justifications**. Each question gives 3 for the correct true/false answer and 2 for the justification—though sometimes a justification will make clear that the true/false answer was a silly error and hence count for more than 2 points.

(a) There is a 2-to-1 correspondence between points on the Bloch sphere and unitary 2×2 matrices.

(b) If the exact number k of solutions in the Grover oracle is known in advance, then finding a solution with probability at least 0.9 requires at most 10 Grover iterations.

(c) If a number a has period r modulo a prime p and period r modulo a different prime q, then a has period r modulo pq.

(d) If U_1 and U_2 are unitary matrices, then $\frac{1}{2}(U_1 + U_2)$ is always a unitary matrix.

(e) If A and B are Hermitian matrices, then $\frac{1}{2}(A + B)$ is always a Hermitian matrix.

(2) (3 + 3 + 12 + 6 + 6 = 30 pts. total)

Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(a) Is B unitary?

(b) Is B Hermitian?

(c) Can you (nevertheless) find eigenvalues λ_1 and λ_2 and unit vectors $\mathbf{u_1}$ and $\mathbf{u_2}$ such that $B\mathbf{u_1} = \lambda_1\mathbf{u_1}$, $B\mathbf{u_2} = \lambda_2\mathbf{u_2}$, and $\langle \mathbf{u_1}, \mathbf{u_2} \rangle = 0$?

(d) Verify that $B = \lambda_1 \left|\mathbf{u_1}\right\rangle \left\langle\mathbf{u_1}\right| + \lambda_2 \left|\mathbf{u_2}\right\rangle \left\langle\mathbf{u_2}\right|$.

(e) Use the above to find a matrix A such that $A^2 = B$.

(scratchwork page)

(3) (12 + 3 + 6 = 21 pts. total)

Make a quantum circuit C on three qubits by placing the following gates:

1. Hadamard gates on lines 1 and 2 only.
2. A CNOT gate with control on line 1 and target on line 3.
3. A CZ gate between lines 1 and 2.
4. Hadamard gates only on lines 1 and 2 again.

(a) Presuming that the qubits are all initialized to $|0\rangle$, calculate the resulting pure state on three qubits. Any mode of calculation is fine: "mazes," matrices, or using the symbolic index-based formula for the Hadamard transform.

(b) Among the 3-bit binary strings abc that can result from a measurement of all qubits, is bit c a function of a and b? Justify your answer briefly.

(c) As we saw in Deutsch's Algorithm, the CNOT gate is the function-oracle of the identity function $f(0) = 0$, $f(1) = 1$. It sure looks like the third qubit is depending only on the first qubit, as the CZ gate comes "chronologically after" the CNOT gate, and no other gate touches qubit 3. Explain from your answers to (a,b), however, why this is not true.

(If you are not confident in your answer to (a), you may answer parts (b) and (c) using the state $\frac{1}{2}(|000\rangle + |001\rangle + |110\rangle - |111\rangle)$ instead—but note, the answers may be different and (c) might not have the same motivation.)

(scratchwork page)

(4) (24 pts. total—end of exam)

Let $G = (V, E)$ be any undirected graph with self-loops allowed. Suppose G has a self-loop at some node u. Let us make a new graph $G' = (V', E')$ with $|V'| = |V| + 2$ by adding two new nodes v and w. Then we *replace* the self-loop by a triangle of edges (u, v), (v, w), and (w, u). Let C and C' be the graph-state circuits constructed in the usual way from G and G', respectively. Show that

$$\langle 0^{n+2} \mid C' \mid 0^{n+2} \rangle = a \langle 0^n \mid C \mid 0^n \rangle$$

for some constant a, so that in particular, G is "net-zero" $\iff$ G' is net-zero. Is the constant a equal to 1, $\frac{1}{\sqrt{2}}$, $\frac{1}{2}$, or $\frac{1}{4}$?