CSE439, Fall 2025 Problem Set 2 Due Thu. 9/25, 11:59pm

Reading: For next week, read the rest of Chapter 5, read Chapter 6, and yes also read Chapter 7. You may find that Chapter 7 actually recapitulates a lot of stuff; for instance, I have already made the boxed point in section 7.6 about entanglement. Also read the Chapter 7 end notes, and for Yogi Berra, see https://yogiberramuseum.org/about-yogi/yogisms/.

——————-*Assignment 2, due Thu. 9/25 "midnight stretchy" on CSE Autolab*——————-

(1) (a) Compute the following tensor products of basic quantum gate matrices seen in lecture: (i) $\mathbf{Z} \otimes \mathbf{S}$, (ii) $\mathbf{S} \otimes \mathbf{Z}$, (iii) $\mathbf{S} \otimes \mathbf{T}$, (iv) $\mathbf{H} \otimes \mathbf{X}$, and (v) $\mathbf{E} = \mathbf{H} \otimes (\mathbf{H} \cdot \mathbf{X})$. (The expression for $\mathbf{E}$, which we've given a name just for this question, includes an ordinary matrix product inside the parentheses.)

(b) Now compute $\mathbf{E}\,|{+}{+}\rangle$, that is $0.5\mathbf{E}[1,1,1,1]^T$, *without* doing any 4×4 matrix operations. (Well, you may do the 4×4 matrix-on-vector multiplication to check your work, but you must show how you can get the answer via operations on 2×2 matrices and between length-2 vectors only. 9 pts. total for (a), (9) for (b), making 18 on this problem.)

(2) (a) Design a 4×4 unitary matrix $\mathbf{U}$ such that $\mathbf{U}e_{00} = \frac{1}{\sqrt{50}}[1,2,3,6]^T$. For a strategy hint, note how this vector comes from problem (3) of assignment 1. (9 pts.)

(b) Now design a 4×4 unitary matrix $\mathbf{V}$ such that $\mathbf{V}\,|{+}{+}\rangle = |{+}{+}\rangle$ and $\mathbf{V}\,|{-}{+}\rangle = |{-}{-}\rangle$. OK, whereas the notation e_{00} in (a) is interchangeable with $|00\rangle$, I don't know any simple "non-Dirac" names for the three states mentioned here. But as a reminder:

- $|{+}{+}\rangle = |+\rangle \otimes |+\rangle = \frac{1}{2}([1,1]^T \otimes [1,1]^T) = \frac{1}{2}[1,1,1,1]^T$.
- $|{-}{+}\rangle = |-\rangle \otimes |+\rangle = \frac{1}{2}([1,-1]^T \otimes [1,1]^T) = \frac{1}{2}[1,1,-1,-1]^T$.
- $|{-}{-}\rangle = |-\rangle \otimes |-\rangle = \frac{1}{2}([1,-1]^T \otimes [1,-1]^T) = \frac{1}{2}[1,-1,-1,1]^T$, and also
- $|{+}{-}\rangle = |+\rangle \otimes |-\rangle = \frac{1}{2}([1,1]^T \otimes [1,-1]^T) = \frac{1}{2}[1,-1,1,-1]^T$.

There are several ways to do this by strategy rather than trial-and-error. One is to consider what the **CNOT** gate does on the standard basis and try to apply "change-of-basis" ideas you may have seen in a previous course. Or you may consider whether permuting the underlying classical co-ordinates might help. Or you can try working out what $\mathbf{V}$ must do on certain linear combinations of $|{+}{+}\rangle$ and $|{-}{+}\rangle$ and maybe other vectors. (12 pts.)

(c) Finally show that such a matrix $\mathbf{V}$ *cannot* be a tensor product of two 2×2 matrices—because if it were, the resulting action on the separate qubits would be self-contradictory. (6 pts., for 27 tota.l)

(3) Show, however, that there is *no* unitary matrix $\mathbf{W}$ such that $\mathbf{W}\,|00\rangle = |00\rangle$, $\mathbf{W}\,|10\rangle = |11\rangle$, $\mathbf{W}\,|{+}0\rangle = |{+}{+}\rangle$ and $\mathbf{W}\,|{-}0\rangle = |{-}{-}\rangle$. Here $|{+}0\rangle$ means $|1\rangle \otimes e_0 = \frac{1}{\sqrt{2}}([1,1]^T \otimes [1,0]^T) =$

$\frac{1}{\sqrt{2}}[1,0,1,0]^T$, and $|{-0}\rangle$ similarly equals $\frac{1}{\sqrt{2}}[1,0,-1,0]^T$. The intent is clear: **W** wants to copy the state of its first qubit over its zeroed-out second qubit. (This will give an even stronger proof of the **no-cloning theorem** than the one to come in Tuesday's lecture. For a hint, you need only three of those four equations to reach a contradiction, if **W** would exist. Use the principle of linearity. 18 pts.)

(4) Design a graph-state circuit C such that given the all-zero state e_{000} asd input, C produces the state

$$\Phi = \frac{1}{2}(e_{000} - e_{001} + e_{101} + e_{111}).$$

Here C must begin and end with $\mathbf{H}^{\otimes 3}$ and is allowed only **CZ** and simple **Z** gates between those two banks of Hadamard gates. (A **Z** gate represents a self-loop at a vertex, rather than an edge ebtween two vertices of the graph.) Note that $-\Phi = \frac{1}{2}(-e_{000} + e_{001} - e_{101} - e_{111})$ is considered to be the same quantum state as Φ, but $\frac{1}{2}(e_{000} + e_{001} + e_{101} - e_{111})$ is really a different state. You are welcome to use a quantum circuit simulator such as those shown in class and do trial-and-error, but there are also strategic ways that track which standard basis vector(s) get negated when a **Z** or **CZ** gate is put into a certain place in the graph. Please show or explain how you got your answer regardless—this may include pasting a snip or screenshot from the simulator. (In Dirac notation, we want to build a graph-state circuit C such that

$$C\,|000\rangle = \frac{1}{2}(|000\rangle - |001\rangle + |101\rangle + |111\rangle).$$

18 pts., for 81 total on the set.) **Whoops! The state was meant to be:**

$$\Phi = \frac{1}{2}(e_{000} - e_{010} + e_{101} + e_{111}) = \frac{1}{2}(|000\rangle - |010\rangle + |101\rangle + |111\rangle).$$