CSE439, Fall 2025   **Problem Set 3**   **Due Thu. 10/2, 11:59pm**

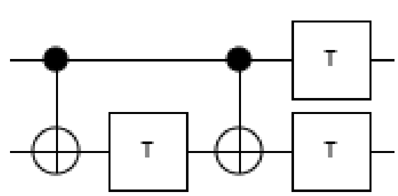
Reading: For next week, finish Chapter 7 and read Chapter 8. This is the last chapter that will be covered by the **First Prelim Exam**, which will be held *in class period* on **Thursday, October 9**. The doman of the exam will be homeworks through this one and factual material such as found on multiple-choice or true/false-type questions, from chapters 1–8 (most parts covered, except sections 6.5 and 6.7 were skipped for now) and supplementary statements in lecture notes. Dirac notation and standard linear algebra notation may be freely mixed in your answers, e.g. $|010\rangle$ and e_{010} meaning the same thing, with the latter being the same as "e_2" when the standard basis vectors in $\mathbb{C}^8$ (or in $\mathbb{R}^8$) are numbered $e_0, \dots, e_7$. I have, however, used Dirac notation uniformly on this assignment sheet.

——————-*Assignment 3, due Thu. 10/2 "midnight stretchy" on CSE Autolab*——————-

(1) A non-standard but sensible notation for the **CZ** gate is "$(-1)^{AND}$." When we separate its two-qubit input signal into its standard-basis components, the component for $|11\rangle$ is multiplied by -1 since the AND is true. The AND is false for the other three components, so nothing happens (i.e., multiplying by $(-1)^0 = 1$). Interestingly, we can also think of **CZ** as "$(-1)^{NAND}$." That is true on $|00\rangle$, $|01\rangle$, and $|10\rangle$. The matrix for "$(-1)^{NAND}$" has $-1, -1, -1, 1$ on the main diagonal. But it is equivalent to **CZ** because it just multiplies the whole thing by -1. Whereas, "$(-1)^{OR}$" with matrix $diag(1, -1, -1, -1)$ is meaningfully different because it is not simply a scalar multiple of the other two 4×4 matrices.

We can say similar things about the **CS** gate: It carries out "i^{AND}." The matrix of "$(-i)^{NAND}$" is equivalent because it multiplies the matrix of **CS** by $-i$. But "i^{OR}" $= diag(1, i, i, i)$ is not so directly equivalent—though it is equivalent in a broader sense that the rest of this problem develops. There's also $(-i)^{AND}$, which differs from **CS** only in having $-i$ in the lower-right corner in place of i. This is the matrix of $\mathbf{CS}^*$. $\mathbf{CS}^*$ is tantamount to **CS** in many respects; note also that $\mathbf{CS}^* = \mathbf{CS}^3$ and vice-versa.

(a) First, verify that the following circuit computes "i^{OR}." You can multiply its 4×4 matrices together, remembering right-to-left versus left-to-right, or show that it agrees on the four standard basis vectors and use the principle of linearity.



(One caveat: If you use a simulator to check this, beware that it *could* show you a unit-scalar multiple of the output on a standard basis state. So try inputs like $|{+}{+}\rangle$ or $|{-}{-}\rangle$ as well where the whole linear combination is outputted, to see that it agrees with the input one.)

(b) Now add gates from the set of single-qubit *Clifford* gates—which allows unit scalar multiples and $\{\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{H}, \mathbf{S}\}$ but not any more **T** gates—to make the resulting circuit compute

CS exactly. It's also fine if you get $\mathbf{CS}^*$ instead. (You don't need to use all of those gates. Using a simulator app is OK, but based on your answer to (a), you should be able to prove this fact. You may not use any 2-qubit gates either. The buzzword is that **CS** and the matrix in part (a) are *locally Clifford equivalent*, meaning that one is obtained from the other by adding single-qubit Clifford gates only.)

(c) Which implication does this prove: (i) $\{\mathbf{H}, \mathbf{CNOT}, \mathbf{T}\}$ is universal $\implies$ $\{\mathbf{H}, \mathbf{CS}\}$ is universal, or (ii) the other way around, i.e., $\{\mathbf{H}, \mathbf{CS}\}$ is universal $\implies$ $\{\mathbf{H}, \mathbf{CNOT}, \mathbf{T}\}$ is universal? (Never mind that lectures did not *prove* the universality of any gate set. 9+9+3 = 21 pts.)

(2) Create quantum circuits C such that $C\,|000\rangle$ produces the following quantum states. This time C need not be a graph-state circuit—in particular, it does not need to close with three Hadamard gates, and you may use Toffoli and **CCZ** gates (in any orientation) and swap gates (between any pair of qubits) and even controlled swap gates. You may use a circuit simulator to design and test your circuits. Note that (b) is the originally-given state from Assignment 2, and likewise (a) *cannot* be done with a graph-state circuit. Ideas from 1(b) may come in handy too. ($9 + 6 + 9 = 24$ pts.)

(a) $\frac{1}{2}(|000\rangle + |001\rangle + |101\rangle + |111\rangle)$.

(b) $\frac{1}{2}(|000\rangle - |001\rangle + |101\rangle + |111\rangle)$.

(c) $\frac{1}{2}(|001\rangle - |011\rangle + |100\rangle + |110\rangle)$.

(3) (a) Draw the graph-state quantum circuit C_G for the graph $G = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{(1, 2), (2, 3), (1, 1), (3, 3)\}$. That is, G is the undirected graph on three nodes with two edges and loops on each end (not in the middle). You may use a snip from Davy Wybiral's applet or *Quirk* or another simulator showing the circuit instead.

(b) Use a "maze diagram" to compute $\langle 000|\, C_G\, |000\rangle$—at least to tell whether this amplitude is zero. Recall from lecture that the "wavefront" after the initial stage $\mathbf{H}^{\otimes 3}$ will have all-positive "mice," and the final $\mathbf{H}^{\otimes 3}$ stage going back up to $|000\rangle$ will not change the signs of any mice after they get there. So you only need to draw the middle section of the diagram for the two **CZ** gates and the two **Z** gates (which can be in any order), then count how many mice end up still positive and how many end up negative.

(c) Now make G' by adding a self-loop at the middle too. What happens to $\langle 000|\, C_{G'}\, |000\rangle$? Tweak your trace from part (b) to show the answer. Can you tell *from the maze diagrams* whether the full state $C_{G'}\,|000\rangle$ changed at all compared to $C_G\,|000\rangle$ in (b)? (21 pts. total, for 66 on the problem set)