CSE491/596 Lecture Wed. 9/23/20   Myhill-Nerode Theorem and Regular Languages

The theorem was proved in 1958 by John Myhill and Anil Nerode while they were at the University of Chicago. Myhill joined the UB Math Department in 1966 and stayed until his death in 1987. Nerode joined Cornell in 1959 and is still active---he is the longest serving professor there and directed the Cornell Mathematical Sciences Institute in the 1980s when I was fortunate to earn an MSI postdoc there. I was hosted in the Computer Science Department by Juris Hartmanis. Here is a post yesterday from Professor Nerode's Facebook stream:

Anil Nerode
1d ·
300 under development?
FT.COM
How close is a coronavirus vaccine? | Free to read
Out of 300 candidates, nine are in phase 3 trials, with some hoping for...

We have proved only one direction. The whole theorem says:

Theorem: A language L is regular $\iff$ all PD sets for L are finite.

We've proved that if L has an infinite PD set, then L is not regular. This is the $\Longrightarrow$ direction, though it may sound like the reverse. It is the contrapositive of the $\Longrightarrow$ direction. To complete the equivalence, we need to prove the $\Longleftarrow$ direction.

Proof: All PD sets for L are finite is the same as saying the equivalence relation $\sim_L$ has only finitely many equivalence classes. Take Q to be the set of equivalence classes. For any string $x \in \Sigma^*$ (where Σ is understood to be the alphabet that L is "over"), there is exactly one equivalence class R_x to which it belongs. Note that R_ϵ is an equivalence class, thus a member of Q, and it will serve as the start state s of the DFA M we are building. Next define

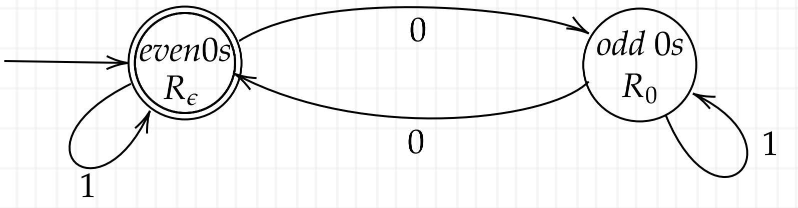
$$F = \{R_x : x \in L\}.$$

Note that even though L may be infinite, F can be finite because R_x and R_y can coincide---indeed, will coincide whenever $x \sim_L y$. Indeed, F must be finite, because F is a subset of Q which is finite by the premise of $\Longleftarrow$. Finally, we define δ by the rule

$$\delta(R_x, c) = R_{xc} .$$

For this to be "well defined" we need to show that it depends only on the equivalence class, not on any x that happens to represent it. So suppose $y \sim_L x$, i.e., that y also belongs to R_x, so that $R_y = R_x$. We need to show that $\delta(R_y, c) = R_{xc}$ too. This follows if R_{yc} is the same as R_{xc}. And justifying this is left as a study guide. Then $M = (Q, \Sigma, \delta, s, F)$ is a legal DFA. And $L(M) = L$ because M hits its accepting states exactly on the strings x that belong to L. Thus L is regular. ☒

We can use this $\Longleftarrow$ direction to help us understand regular languages and build DFAs for them. Here is a simple example with $\Sigma = \{0,1\}$. Consider $L = \{x \in \Sigma^* : \#0(x) \text{ is even}\}$. [Note: In the actual lecture, I quantified $\#1(x)$ instead, but this works better with the example that followed.] Then $x \sim_L y$ iff the numbers of 0s in x and y are both even or both odd. Hence the relation $\sim_L$ has just two equivalence classes. The DFA M is basically one we have already seen:



$S = \{\epsilon, 0\}$ is a PD set of size 2, and no PD set can be bigger.

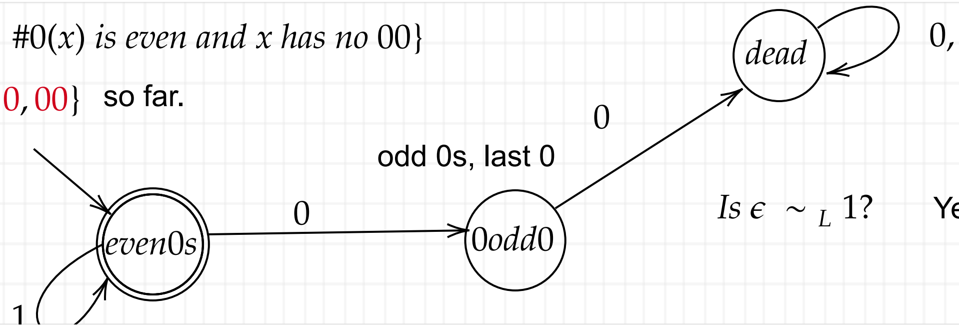
Now let's try a trickier example by conjoining "even 0s" with another condition of not having 00 as a substring:

$$L = \{x : \#0(x) \textit{ is even and } x \textit{ has no } 00\} \qquad (1)$$

[In regular expression terms, L equals $\left(1^*01^*0\right)^*1^* \setminus (0+1)^*00(0+1)^*$ but set-minus $\setminus$ is not a native regular operator so that doesn't help us even build an NFA, let alone a DFA, to accept L. So let's ignore this attempt and try using (1) to build a DFA M by "MNT-enlightened trial and error."] We know that $\epsilon \in L$, so the start state will be accepting, and that 0 and 00 are both not in L. Indeed, 00 causes a "dead condition" because no string beginning with 00 can possibly belong to L, so it should go to a dead state. That gives us part of the machine:

$L = \{x : \#0(x) \textit{ is even and } x \textit{ has no } 00\}$

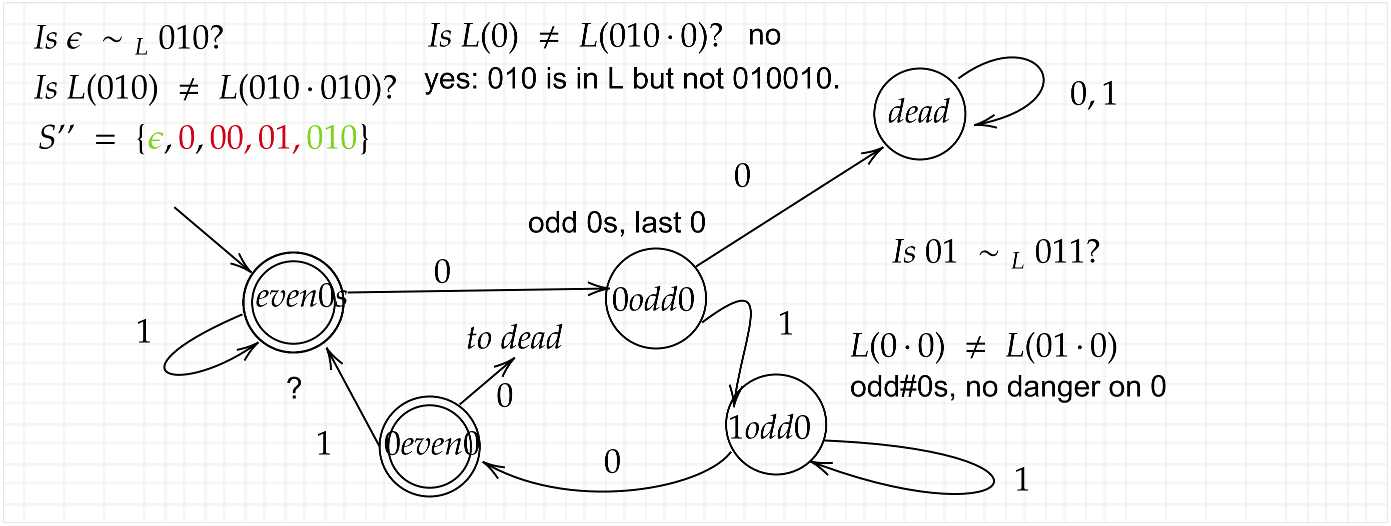
$S = \{\epsilon, 0, 00\}$ so far.



$Is\ \epsilon \sim_L 1?$ Yes

How about the string 1? It can still be a loop at the start state. At the left end of a string it makes no difference to having a possible 00, so $1 \in R_\epsilon$. But what about the loop on 1 which we had at the "odd" state? Can we still direct it back to that state? It is equivalent to ask whether $0 \sim_L 01$. To see why not, consider $x = 0$ and $y = 01$. Take $z = 0$. Then $xz = 00$ is not in L but $yz = 010$ is indeed in L, because the 1 helped us avoid a 00. For the same reason, $01 \nsim_L 00$, and clearly $01 \nsim \epsilon$ because ϵ is in L and 01 is not (technically, they are distinguished by $z = \epsilon$). Thus $S' = \{\epsilon, 0, 00, 01\}$ is a PD set of size 4, and so we need a fourth state to process it to. Now, what about that string 010? It is in L, but does it belong to R_ϵ?

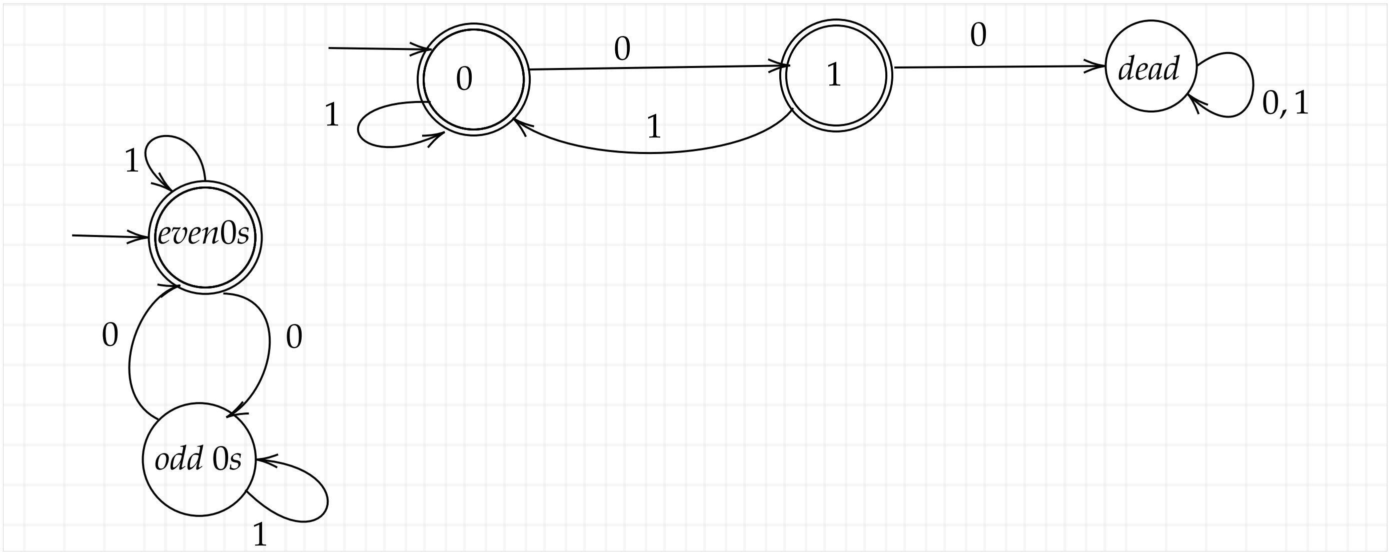
It does not, but finding a string z such that $L(\epsilon \cdot z) \neq L(010 \cdot z)$ is not so fast. We need to activate the "no 00" condition by making z begin with 0, but then we need another 0---but not right away---to make $z \in L$. Thus $z = 010$ is the shortest distinguishing string. This gives us:



Is $\epsilon \sim_L 010$?
Is $L(010) \neq L(010 \cdot 010)$?
$S'' = \{\epsilon, 0, 00, 01, 010\}$
Is $L(0) \neq L(010 \cdot 0)$? no
yes: 010 is in L but not 010010.
dead
0, 1
0
odd 0s, last 0
even0s
0odd0
0
1
to dead
1
0
?
1
0even0
1odd0
0
1
Is $01 \sim_L 011$?
$L(0 \cdot 0) \neq L(01 \cdot 0)$
odd#0s, no danger on 0

So we wound up needing 5 states. Is that enough? Well, can we complete the machine with arcs from the "even 0s, last char 0" state? Clearly 0 goes to $dead$, and 1 must go to an accepting state. If 1 can go to the start state, then we're done. Can it? Yes---by similar reasoning to putting a loop on 1 at the start state. So M is done and S'' is a largest possible PD set.

There is another kind of reasoning we could have done. L is the $\cap$ of two languages represented by the 2-state DFA above and the following simple 3-state DFA for the "no substring 00" condition:



[Rest of lecture added, will pick up briefly.]
Doing the Cartesian Product construction seems to suggest the final DFA will have $2 \times 3 = 6$ states. But the operation is intersection, so the "dead" condition in the upper DFA knocks-on to make the whole third column dead in the product machine. Since you don't need two separate dead states, the number goes down to 5 after all. It is a good exercise to carry out the construction and verify that you get the same 5-state DFA as above. In fact, you *must* get the same machine, because of the following corollary to the MNT.

Corollary: In the $\Leftarrow$ direction of MNT, the DFA you get not only has the least possible number of states, it is unique. Hence, every regular language has a *unique minimum-size DFA*. ☒

Putting a checkbox in the corollary statement signifies that we've already essentially proved it. The notes by Debray prove instead that every DFA can be reduced to a unique minimum one via the *DFA minimization algorithm*. The algorithm is interesting for its own sake as IMHO the easiest example of "dynamic programming" but for us it is just a "skim". The reasoning of both halves of MNT helps us recognize minimum-size DFA cases, even extreme ones.

Proposition: For each $k \geq 1$, the unique minimum DFA for $L_k = (0+1)^* 1 (0+1)^{k-1}$ has 2^k states.

Proof: Take $S = \{0,1\}^k$. Then S has size 2^k. We claim that S is PD for L_k: Let any $x, y \in S$,
$x \neq y$,
be given. By $x \neq y$, there is some position i (let's number from 1) in which they differ. Take
$z = 0^{i-1}$. Then xz and yz differ in position k from the end, so $L_k(xz) \neq L_k(yz)$. This pproves the claim, so the consequence is that any DFA M_k such that $L(M_k) = L_k$ needs at least 2^k states. Well, we can build a correct M_k of that size by having one state q_w for each possible combination w of last k bits read (treating an initial small string like 10 as if it had $k-2$ leading 0s) and defining $\delta(q_{bv}, c) = q_{vc}$.
The final states are q_w for those w that begin with 1: since $|w| = k$, this 1 is in the kth position from the right. So this M_k is the unique minimum DFA for L_k. ☒

Note that the NFA N_k from an earlier lecture only needs $k+1$ states. Thus this also demonstrates cases where the NFA-to-DFA construction has an *unavoidable* "exponential explosion." Furthermore, the regular expression for L_k in the proposition statement (call it r_k) needs only $12 + \log_2(k)$ symbols, the log part for the bits in the number $k-1$. This is an exponential step **down** in size. The upshot is that NFAs can sometimes be exponentially more **succinct** than DFAs, and regular expressions (with numerical powering) can be even more succinct in some cases.