CSE 491/816 | lecture Fri 10/6 | Fall 2023

Decidable and Undecidable Languages / Decision Problems

"G&J Format"

Name of Problem	Non-Emptiness for DFAs
INSTANCE	INST: A DFA $M = (Q, \Sigma, \delta, s, F)$ encoded as a (binary) string in some natural way)
QUESTION	QUES: Is $L(M) \neq \emptyset$? i.e., is there a string that M accepts?

The language of a problem is always the set of legal instances for which the answer is yes.

$\{\underset{\text{DFA}}{M} : L(M) \neq \emptyset\}$

"Sipser Naming Scheme"

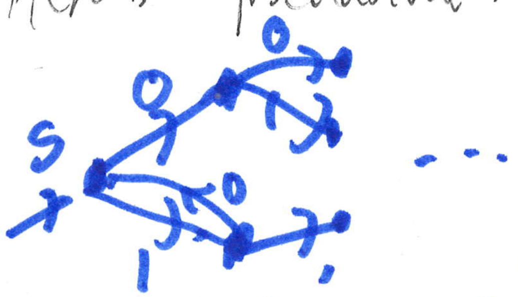
[Type of Problem] subscripted by Objects it applies to

non-emptiness NE_{DFA} — This also is a name for the language

Theorem: NE_{DFA} is {solvable / decidable} in polynomial time.

Proof: Here is a pseudocode sketch of a poly-time decision procedure.

Helpful Picture M =



. . .

① f_1
① f_2

Algm: Breadth First Search.
$G = (V, E)$
$n = |V|$ time $\tilde{O}(n^2)$
$N = |E|$ time $\tilde{O}(N)$

BFS runs in $n^{O(1)}$ time. ∎

Now how about the problem NE_{NFA}?
Same algm, same running time.

Lang. = {NFAs N : $L(N) \neq \emptyset$}
s →ε q OK.

How about $ALL_{DFA} = \{\text{DFAs } M = (Q, \Sigma, \delta, s, F) : L(M) = \Sigma^*\}$

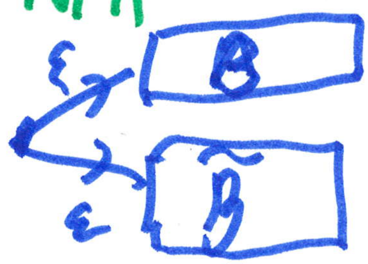
Is this in **P**?

Alg^m:
- Given M, first build $M' = (Q, \ldots Q \setminus F)$
- Run the prev. algm A on M'
- Answer yes iff A said no, and vice versa.

Fact: $L(M) = \Sigma^*$ $\iff$ $L(M') = \emptyset$.

How about $ALL_{NFA} = \{\text{NFAs } N : L(N) = \Sigma^*\}$?

what if $N =$



Fact: ALLNFA is NP-hard! Hence unlikely to be in P.

Converting N to a DFA M and using ALLDFA on M is a decision procedure, just not in polynomial time.

$ALL_{NFA} \in REC$, $\in^? P$ not known.

How about $NE\cap_{DFA}$: INST: Two DFAs M_1, M_2
QUES: Is $L(M_1) \cap L(M_2) \neq \emptyset$?

Alg^m: Build M_3 = the Cartesian Product of M_1 and M_2 with $F_3 = \{(p,q) : p \in F_1$ **&&** $q \in F_2\}$

Then $L(M_3) = L(M_1) \cap L(M_2)$, so answer is yes $\iff \langle M_3 \rangle \in NE_{DFA}$ ("the code of" M_3)

We can build M_3 in time $\approx n_1 \cdot n_2$ (M_1 has n_1 states, M_2 has n_2) then run BFS on M_3. $\therefore$ $n^{O(1)}$ time. So it is in P.

- How about $NE\cap_{NFA}$? (In P)
- How about $NE\oplus_{DFA} = \{\langle M_1, M_2 \rangle : L(M_1) \triangle L(M_2) \neq \emptyset\}$? In P: Build M_3' using XOR, run BFS.
- How about $NE\oplus_{NFA}$? (NP-Hard!) Fix N_1 to acc. Σ^* $N_1 =$  0,1 Then $L(N_1) \triangle L(N_2) \neq \emptyset$ $\iff L(N_2) \neq \Sigma^* \iff \langle N_2 \rangle \notin ALL_{NFA}$