The first half of lecture finished the time and space hierarchy theorems.

Log Space Reducibility $\leq^m_\log$ Finer than $\leq^m$ since L$\subseteq$P.

Log Space Computable Functions

Technical Point: If $M$ and $N$ belong to NL (function logspace) then so does $g(f(x))$.

Issue: If $M$ computes $y = f(x)$ and $M'$ computes $z = g(y)$, we can chain $M$ and $M'$ together but we can't store $y$ within the $O(\log n)$ space.

- If the input tapes of both machine are right-only as well as read-only, then there is no problem: the output $y = f(x)$ of $M$ is streamed to $M'$ computing $g(y) = z$ and never has to be written down.
- If each machine is allowed $r(n)$ left-to-right streaming passes over its input and $y$ is a stream, then the tandem can operate with $r(n)^2$ passes on $x$.
- But if $M'$ can demand to back up to a previous input bit $y_{i-1}$ at any time, then we need to allow $M$ to be restarted arbitrarily many times. This can be implemented by storing the current demand-bit $i$ on another log-sized tape.

Therefore $\leq^m_\log$ reduction are transitive: $A \leq^m_\log B \leq^m_\log C \Rightarrow A \leq^m_\log C$.

In fact, every $\leq^m$ and $\leq^m_\log$ reduction shown in the course has actually been a $\leq^m_\log$ reduction or even the sharper one pass streaming kind.
• All the NP-completeness results we’ve shown have been valid under $\leq \frac{\text{log}}{m}$.

• **GAP** is complete for **NL** under $\leq \frac{\text{log}}{m}$.

• The language **CVP** of the **Circuit Value Problem**: given a Boolean circuit $C_n$ and an input $x \in \{0, 1\}^n$, is $C_n(x) = 1$? is complete for **P** under $\leq \frac{\text{log}}{m}$.

• The language **TQBF** of true **quantified Boolean formulas** is complete for **PSPACE** under $\leq \frac{\text{log}}{m}$.