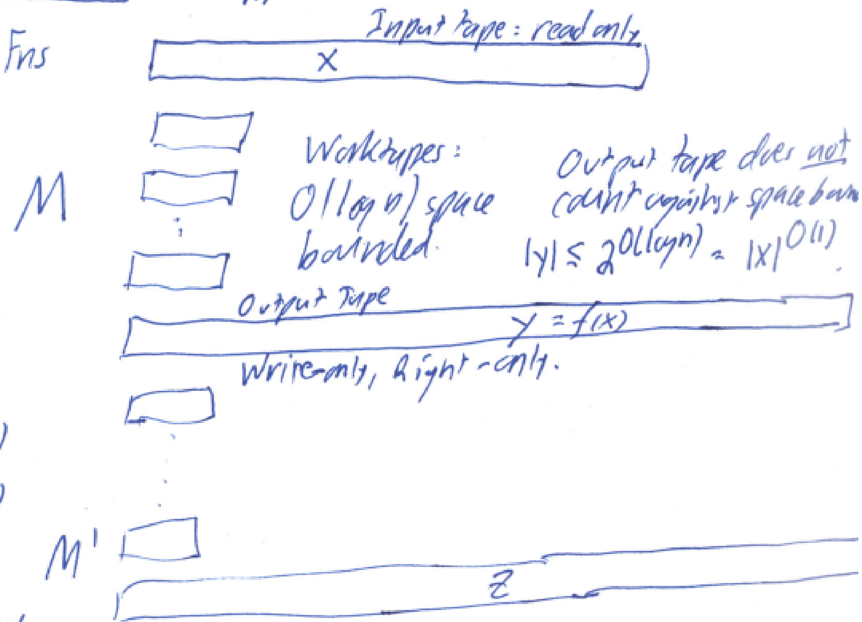
CSE491/596 Lecture Friday, Nov. 13: Completeness Under Logspace Reductions

[The first half of lecture finished the time and space hierarchy theorems.]

Log Space Reducibility $\le_m^{log}$ Finer than $\le_m^p$ since $L \subseteq P$.

Log-Space Computable Fns

Technical Point: If f and g belong to FL (function logspace) then so does $g \circ f$. Issue: If M computes $y = f(x)$ and M' computes $z = g(y)$ we can "chain" M and M' together, but we can't store y within the $O(\log |x|)$ space bd.



- If the input tapes of both machine are **right-only** as well as read-only, then there is no problem: the output $y = f(x)$ of M is streamed to M' computing $g(y) = z$ and never has to be written down.
- If each machine is allowed $r(n)$ left-to-right streaming passes over its input and y is a stream, then the tandem can operate with $r(n)^2$ passes on x.
- But if M' can demand to back up to a previous input bit y_{i-1} at any time, then we need to allow M to be restarted arbitrarily many times. This can be implemented by storing the current demand-bit i on another log-sized tape.

Whenever M' wants to move its input head Left, M re-starts from the beginning until it outputs bit $i-1$ of y, which is stored $\rightarrow \square\ y_i$ $\boxed{i}$. If M' moves to $i+1$, M takes however long to output bit $i+1$. All the re-starting is inefficient for time but stays within $O(\log n)$ space.

Therefore $\le_m^{log}$ reductions are transitive: $A \le_m^{log} B \wedge B \le_m^{log} C \Rightarrow A \le_m^{log} C$.

In fact, every $\le_m$ and $\le_m^p$ reduction shown in the course has actually been a $\le_m^{log}$ reduction or even the sharper one-pass streaming kind.

Hallmarks of a $\le_m^{\log}$ Reduction:

- The objects it constructs have an explicit formula. E.g.:
  $G_\phi = (V_\phi, E_\phi)$, $V_\phi = \{x_i, \bar{x}_i : 1 \le i \le n\} \cup \{x_{ij}, \bar{x}_{ij} :$ variable x_i is in clause C_j possibly negated$\}$
  $E_\phi = \{ \cdots \} \cup \{ \cdots \}$ etc.

- The individual items used in building G_ϕ etc. are finite clumps of $O(\log n)$-sized labels such as variable numbers i, clause #s j.

- (In consequence), local features of the target object G_ϕ (or etc.) depend only on local features of the source object (e.g., $C_1 \wedge \cdots \wedge C_m$) or on simple global connections—like copying $\langle M, w \rangle$ or hooking up the B and G nodes in the $3SAT \le_m^{\log} G3C$ example.

- All the NP-completeness results we've shown have been valid under $\le_m^{\log}$.
- **GAP** is complete for NL under $\le_m^{\log}$.
- The language **CVP** of the **Circuit Value Problem**: given a Boolean circuit C_n and an input $x \in \{0,1\}^n$, is $C_n(x) = 1$? is complete for P under $\le_m^{\log}$
- The language **TQBF** of true **quantified Boolean formulas** is complete for PSPACE under $\le_m^{\log}$.