CSE 491/596 Lecture 11/28/22 Fall 2022

How does Nature Calculate? Is Nature Lexical?

"It From Bit"—? (John Wheeler)

The most basic lexical operation is concatenating chars.

$a \cdot b \cdot b \cdot a \neq baba \neq aabb$ Not Commutative.

Which numerical operations can emulate concatenation?
Which ones are not commutative?

How about Matrix Multiplication? $AB \neq BA$ in general.

So we can make $A \cdot B \cdot B \cdot A$, $B \cdot A \cdot B \cdot A$, $A \cdot A \cdot B \cdot B$ $(n \times n)$
and they might not be equal.

Problem: these products are a single $n \times n$ matrix, and it might lose the sequencing, indeed the factors.

¿Nature; Answer? Tensor Product of matrices.

Given $A: m \times n$
$B: p \times q$
make $A \otimes B$

$$\underline{A \otimes B} = \begin{bmatrix} a_{11} B & a_{12} B & a_{13} B & \cdots & a_{1n} B \\ a_{21} B & a_{22} B & a_{23} B & \cdots & a_{2n} B \\ \vdots & & & & \\ a_{m1} B & a_{m2} B & a_{m3} B & \cdots & a_{mn} B \end{bmatrix}$$

$(mp \times nq)$

Tensor Product works for "row vectors" $\approx 1\times n$ matrices and "col-vectors" as $m\times 1$ matrices!

$a=\begin{pmatrix}1\\0\end{pmatrix}$ $b=\begin{bmatrix}0\\1\end{bmatrix}$

$$a\otimes b=\begin{pmatrix}1\begin{pmatrix}0\\1\end{pmatrix}\\0\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}=e_1$$

$$a\otimes a=\begin{pmatrix}1\cdot\begin{pmatrix}1\\0\end{pmatrix}\\0\cdot\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=e_0$$

$$b\otimes a=\begin{pmatrix}0\cdot\begin{pmatrix}1\\0\end{pmatrix}\\1\cdot\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}=e_2$$

$$b\otimes b=\begin{pmatrix}0\begin{pmatrix}0\\1\end{pmatrix}\\1\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix}=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}=e_3$$

$aa = e_0$ $ab = e_1$ $ba = e_2$ $bb = e_3$

Row vectors: $(1\ 0)\otimes(1\ 0)=\big(1\cdot(1\ 0)\quad 0(1\ 0)\big)$
$=(1\ 0\ 0\ 0)$ etc.

Triple Products: $\otimes$ is associative

Say $aaa = a\otimes a\otimes a=(a\otimes a)\otimes a=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\otimes\begin{bmatrix}1\\0\end{bmatrix}=\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}$

And $aaaa \equiv a\otimes a\otimes a\otimes a = a^{\otimes 4}$ = a vector of height 16

$a^{\otimes n}$ = a vector of size 2^n !!

Likewise the tensor representation of any other word of size n.

Matrix example.

Hadamard Matrix

$$H=H_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\qquad H^{\otimes 2}=\begin{bmatrix}1\begin{bmatrix}1&1\\1&-1\end{bmatrix}&1\begin{bmatrix}1&1\\1&-1\end{bmatrix}\\1\begin{bmatrix}1&1\\1&-1\end{bmatrix}&-1\begin{bmatrix}1&1\\1&-1\end{bmatrix}\end{bmatrix}$$

$$=\begin{bmatrix}1&1&1&1\\1&-1&1&-1\\1&1&-1&-1\\1&-1&-1&1\end{bmatrix}$$

(Times $\frac{1}{2}$ to normalize)