(1) Use the Myhill-Nerode proof script to show that the following languages over $\Sigma = \{0, 1\}$ are not regular:

(a) $L_a = \{v1v : v \in \Sigma^*\}$.
(b) $L_b = \{v1w : v, w \in \Sigma^*, |v| = |w|\}$.
(c) $L_c = \{v1w : \#0(v) = \#0(w)\}$.

The points total 18 for all three languages together; you need not have three separate answers depending on how you organize things. Recall from lecture that $\#0(x)$ means the number of 0s in the string $x$.

**Answer:** For all three, take $S = 0^*1$. [$S = 0^*$ is also fine; then prepend the needed 1 to $z$.] Clearly $S$ is infinite. Let any $x, y \in S$, $x \neq y$, be given. Then we can write $x = 0^m1$ and $y = 0^n1$ where $m, n \geq 0$ and $m \neq n$. [We could assume without loss of generality that $m < n$ but will not need to do so.] Take $z = 0^m$. Then observe:

(a) $xz = 0^m10^m$ is in $L_a$ (with $v = 0^m$) but $yz = 0^n10^m \notin L_a$ because there is only one 1 and the strings to its right and left are not the same;

(b) $xz = 0^m10^m \in L_b$ because the strings to the left and right of the 1 have the same length, but $yz = 0^n10^m \notin L_b$ because there they do not;

(c) $xz = 0^m10^m \in L_c$ because the strings to the left and right of the 1 have the same number of 0s, but $yz = 0^n10^m \notin L_b$ because they have different numbers of 0s.

Since $x, y \in S$ are arbitrary, $S$ is a PD set for each of $L_a, L_b, L_c$, and since $S$ is infinite, all three languages are non-regular by the Myhill-Nerode Theorem.

(2) Now do $L_d = \{v1w : \#1(v) > \#0(w)\}$. (Be careful to note that the 1 shown could be anywhere in the given string $x$ when you break it as $x = v1w$. 18 pts.)

Take $S = 1^+$. Clearly $S$ is infinite. Let any distinct $x, y \in S$ be given. Then we can write $x = 1^m$ and $y = 1^n$ where $m, n \geq 1$ and without loss of generality, $m < n$. Take $z = 0^{m-1}$. This is “legal” because $m \geq 1$, since we used $S = 1^+$ rather than $1^*$. Then $xz = 1^m0^{m-1}$ does not belong to $L_d$ because we need one of the 1s to be the 1 shown in the definition and $w$ must be all of $0^{m-1}$. This leaves only $m - 1$ of the 1s to use in $v$, which is not greater than $\#0(w)$. But $yz = 1^n0^{m-1}$ does belong to $L_d$ because we get $v = 1^{n-1}$ and $w = 0^{m-1}$, and $n - 1 > m - 1$ because $m < n$. Thus $S$ is an infinite PD set for $L_d$, so $L_d$ is not regular by the Myhill-Nerode Theorem.

*Just as good an answer:* Take $S = 1^*$ after all. Clearly $S$ is infinite. Let any distinct $x, y \in S$ be given. Then we can write $x = 1^m$ and $y = 1^n$ where $m, n \geq 0$ and without loss of generality, $m < n$. Take $z = 10^m$. Then $xz = 1^m1 \cdot 0^m$ does not belong to $L_d$ because we only get $\#1(v) = \#0(w)$, not greater. But $yz = 1^n10^m$ does belong to $L_d$ because $n > m$. Thus $S$ is an infinite PD set for $L_d$, so $L_d$ is not regular by the Myhill-Nerode Theorem.
(3) Design a two-tape Turing machine $M$ such that $L(M) = L_c$ above and $M$ runs in $O(n)$ time. Pseudocode is fine provided it is detailed enough to demonstrate running in $O(n)$ time.

Answer: On any input $x \in \{0, 1\}^*$, do the following:

(i) Do a left-to-right pass and push an $X$ for every second 0 read.

(ii) If at the end the number of 0s is odd, immediately reject. If there were no 0s, then accept if and only if at least one 1 was read.

(iii) Else, we know $\#0(x) = 2k$ for some $k \geq 1$. So begin a right-to-left pass on tape 1 and pop an $X$ for every 0 read.

(iv) When the last $X$ is popped, the head will be on the $k$th 0 from the right end. Finally accept $x$ if and only if the character to its left is a 1.

The correctness is clear from seeing there are $k$ 0s to the right of the 1 found at the end, and the other $k$ 0s are to its left. Whereas, if the char to the left is a 0, this means there is no 1 to meet the condition defining $L_c$. The time taken is at most twice the lengths of $x$, so this runs in $O(|x|)$ time. [Moreover, this $M$ is a deterministic PDA, and all DCFLs belong to linear time.]