


Defⁿ: An NFA N can process a string $x \in \Sigma^*$ from state p to state q if there is a sequence $(q_0, u_1, q_1, u_2, q_2, \dots, q_{m-1}, u_m, q_m)$ such that:

- $q_0 = p$ and $q_m = q$,
- for all $i, 1 \leq i \leq m, (q_{i-1}, u_i, q_i)$ is a legal instruction possibly with ϵ -transitions

and $u_1 u_2 \dots u_m = x$. [By convention, $|x| = n$, but some u_i can be ϵ so $m > n$ is possible.]

The sequence is called a (valid) computation (path).

Defⁿ: $L_{pq} = \{x \in \Sigma^* : N \text{ can process } x \text{ from } p \text{ to } q\}$, and finally formally:
 $L(N) = \bigcup_{q \in F} L_{sq} = \{x \in \Sigma^* : N \text{ can process } x \text{ from } s \text{ to some state in } F\}$.

Example: $N =$ 

Is $x = 1011$ in $L(N)$?
No: $(s, 1, a, 0, a, 1, b, 1, b)$ cannot be validly completed
 $(s, 1, s, 0, s, 1, a, 1, b)$ falls short of state f .

Defⁿ: The concatenation of two languages $A, B \subseteq \Sigma^*$ is defined by

$$A \cdot B = \{xy : x \in A \wedge y \in B\}$$

[Translates English "x" and then "y"]
 compound subject

Does $A \cdot A = \{x \cdot x : x \in A\}$? No = (common error).

$A \cdot B = \{z : z \text{ can be broken as } z = x \cdot y \text{ s.t. } x \in A \wedge y \in B\}$

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$A = \{0, 01\}$ $A \cdot A = \{00, 001, 010, 0101\}$
 (!) (!)

$B = \{0, 10\}$. Also note:
 $A \cdot B = \{0 \cdot 0, 0 \cdot 10, 01 \cdot 0, 01 \cdot 10\}$
 $= \{00, 010, 010, 0110\} = \{00, 010, 0110\}$

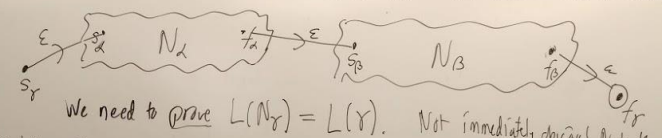
Always $\|A \times B\| = \|A\| \cdot \|B\|$, but $\|A \cdot B\| < \|A\| \cdot \|B\|$ is possible.

Picking up in the inductive cases of the proof, given regexps α, β and equivalent NFAs N_α, N_β .
 "Series Circuit": could identify $s_r \equiv s_\alpha, f_r \equiv f_\beta$, but the middle ϵ -arc can be necessary.

$\gamma = \alpha \cdot \beta$ is a regexp,

$$L(\gamma) =_{\text{def}} L(\alpha) \cdot L(\beta),$$

and given N_α, N_β , build $N_\gamma =$



We need to prove $L(N_\gamma) = L(\gamma)$. Not immediately obvious! Note the flow.

$$L(N_\gamma) = \{z \in \Sigma^* : N_\gamma \text{ can process } z \text{ from } s_r \text{ to } f_r\}$$

By the particular series form of N_γ , $L(N_\gamma) = \{z \in \Sigma^* : z \text{ has a prefix } x \text{ that gets processed from } s_r \text{ via } s_\alpha \text{ to } f_\alpha, \text{ then the bridge } \epsilon \text{ to } s_\beta, \text{ then a suffix } y \text{ processed from } s_\beta \text{ to } f_\beta \text{ before the } \epsilon \text{ to } f_r\}$.

Induction hypothesis about correctness of the NFAs is used here

$$= \{z \in \Sigma^* : z \text{ can be broken as } z = x \cdot y \text{ st. } x \in L(N_\alpha) \cap y \in L(N_\beta)\}$$

$$= L(N_\alpha) \cdot L(N_\beta)$$

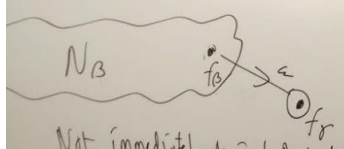
$$=_{\text{def}} L(\gamma) \quad \therefore L(N_\gamma) = L(\gamma),$$

as needed to propagate the induction hypothesis.

$\gamma =$
 $L(\gamma) =$
 $A^* =_{\text{def}} L$
 $= \Sigma$
 $A \cdot B$
 $A \cdot A$
 $A = \{0$
 $B = \{0$
 $A \cap$
 $\|A \times$

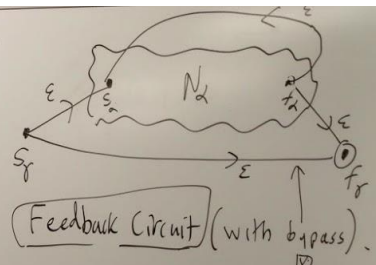
equivalent NFAs N_α, N_β .
 f_β , but the middle ϵ -arc can be necessary.

$\gamma = \alpha^*$ is a regexp. N_γ
 $L(\gamma) = L(\alpha)^*$ defined via



$$A^* =_{\text{def}} A^0 \cup A^1 \cup A^2 \cup A^3 \cup \dots$$

$$= \{\epsilon\} \cup A \cup A \cdot A \cup A \cdot A \cdot A$$



Not immediately obvious! Note the flow. from s_r to f_r

$$A \cdot B = \{z : z \text{ can be broken as } z = x \cdot y \text{ st. } x \in A \cap y \in B\}$$

not gets processed from s_r via s_α ϵ to s_β , then a suffix y processed before the ϵ to f_r .

$$A \cdot A = \{z : z \text{ can be broken as } z = x \cdot y \text{ st. } x \in A \cap y \in A\}$$

$$z = x \cdot y \text{ st. } x \in L(N_\alpha) \cap y \in L(N_\beta) \} \quad B = \{0, 10\}$$

$$A = \{0, 01\} \quad A \cdot A = \{00, 001, 010, 0101\}$$

$L(\gamma)$. $\therefore L(N_\gamma) = L(\gamma)$, as needed to propagate the induction hypothesis.

Also note:

$$A \cdot B = \{0 \cdot 0, 0 \cdot 10, 01 \cdot 0, 01 \cdot 10\}$$

$$= \{00, 010, 010\}$$

Always $\|A \times B\| = \|A\| \cdot \|B\|$, but $\|A \cdot B\| < \|A\| \cdot \|B\|$ is possible.