
Def^n: The concatenation of two languages $A, B \subseteq \Sigma^*$ is $A \bullet B = \{x \cdot y : x \in A \wedge y \in B\}$

Example: $\Sigma = \{0,1\}$

$A = \{0, 01\}$ $B = \{0, 10\}$

$A \cdot B = \{ 0 \cdot 0, \underbrace{0 \cdot 10, 01 \cdot 0}_{\text{same}}, 01 \cdot 10\}$

$= \{00, 010, 0110\}$.

Compare:

$A \times B = \{(0,0), \underbrace{(0,10), (01,0)}_{\text{different as pairs}}, (01,10)\}$

Note:

Always $\|A \times B\| = \|A\| \cdot \|B\|$, but $\|A \cdot B\| < \|A\| \cdot \|B\|$ is possible.

Is $A \cdot A = \{x \cdot x : x \in A\}$? No: $A \cdot A$ above $= \{00, 001, 010, 0101\}$

Rewrite the def^n as: $A \cdot B = \{z \in \Sigma^* : z$ can be broken as $z = x \cdot y$ such that $x \in A \wedge y \in B\}$

$\therefore A \cdot A = \{z : z$ can be broken as $z = x \cdot y$ st. $x \in A \wedge y \in A\}$.

Def^n: $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, and so on. $A^1 = A$.

$0^0 = 1$ $A^0 = ?$ By convention, $A^0 = \{\varepsilon\}$ for all A, even $\emptyset$.

Def^n $A^* = \bigcup_{m=0}^{\infty} A^m = \{\varepsilon\} \cup A \cup A^2 \cup A^3 \ldots$ Kleene Star

Stephen "Clay Knee"

Rejoining the proof of (the first leg of) Kleene's Theorem

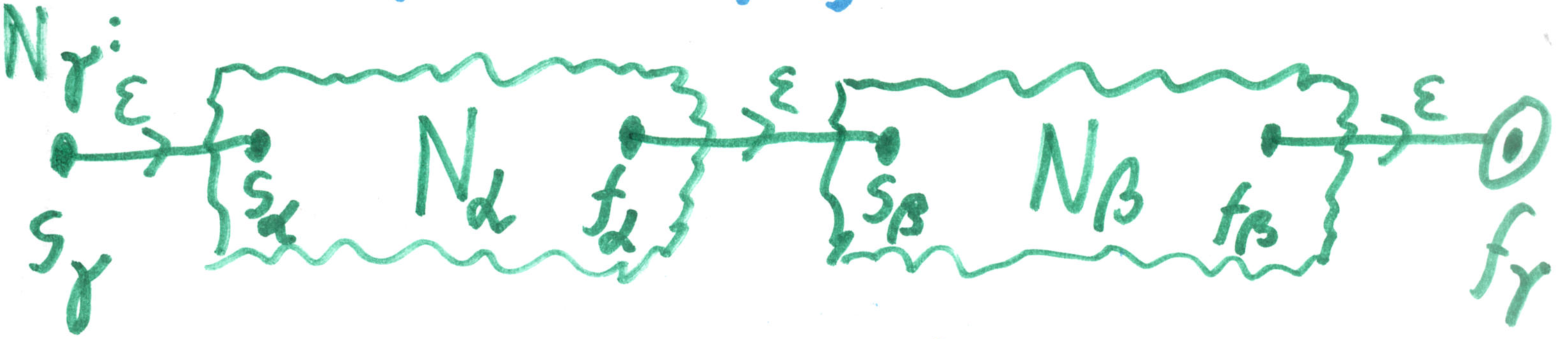
Induction case 5. Given regexps α and β. By Ind. Hyp we may take NFAs N_α, N_β such that

* $L(N_\alpha) = L(\alpha)$
$L(N_\beta) = L(\beta)$

Then $\gamma = \alpha \cdot \beta$ is a regexp,

$L(\gamma) =_{df} L(\alpha) \cdot L(\beta)$, and

we build N_γ from N_α, N_β like so:.



Need to prove: $\underbrace{L(N_\gamma) = L(N_\alpha) \cdot L(N_\beta)}_{\text{This is our first step.}}$ so $= L(\gamma)$

For any string $z \in \Sigma^*$,

$z \in L(N_\gamma) \equiv N_\gamma$ can process z from s_γ to f_γ. By diagram

$\Leftrightarrow$ z can be broken as $z =: xy$ such that N_α can process x from s_α to f_α and N_β can process y from s_β to f_β.

$\Leftrightarrow$ z can be broken as $z =: xy$ such that $x \in L(N_\alpha)$ and $y \in L(N_\beta)$

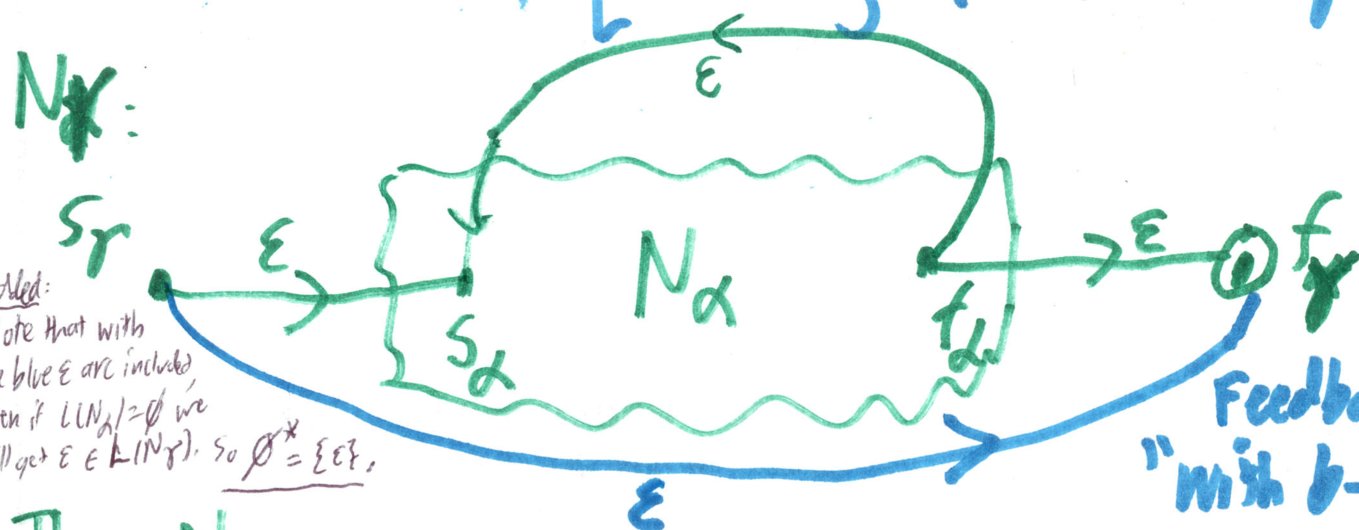
$\Leftrightarrow$ $z \in L(N_\alpha) \cdot L(N_\beta)$. $\therefore \boxed{(\forall z)}: z \in \boxed{L(N_\gamma)} \Leftrightarrow z \in \boxed{L(N_\alpha) \cdot L(N_\beta)}$

$\therefore \boxed{L(N_\gamma)} = \boxed{L(N_\alpha) \cdot L(N_\beta)}$.

By I.H.*, $L(N_\gamma) = L(\alpha) \cdot L(\beta)$, which $= L(\gamma)$

$\therefore$ $L(N_\gamma) = L(\gamma)$ is proved. by defn of $L(\gamma)$.

6. Given just α and N_α s.t. $L(N_\alpha) = L(\alpha)$, we define $\gamma = \alpha^*$ to have the semantics $L(\gamma) = [L(\alpha)]^*$. Build N_γ as:



Added: Note that with the blue ε arc included, even if $L(N_\alpha) = \emptyset$ we still get $\varepsilon \in L(N_\gamma)$. So $\emptyset^* = \{\varepsilon\}$.

Feedback circuit "with bypass"?

Then N_γ can process a string z from s_γ to f_γ iff z can be broken as $z =: \underbrace{z_1 z_2 \cdots z_m}_{\text{substrings, not bits}}$ such that for each i, $1 \leq i \leq m$, N_α can process z_i from s_α to f_α. Since $A^* = \{z : z$ can be broken as $z_1 \cdots z_m$ s.t. for each i, $z_i \in A\}$, we get

$$
\begin{aligned}
L(N_\gamma) &= L(N_\alpha)^* \\
\text{by IH} &= L(\alpha)^* \\
&= L(\gamma). \quad \boxtimes
\end{aligned}
$$

This also completes both the formal inductive definition of regular expressions γ and the proof that some NFA N_γ gives $L(N_\gamma) = L(\gamma)$. $\boxtimes$

Without the blue ε-arc we get $L(N_\alpha)^+ = L(N_\alpha) \cup L(N_\alpha)^2 \cup \ldots$ instead