CSE596 Lecture Mon Sep 17 Fall 2017

Let $L$ be a language we "hope" is regular.
Let $x$ and $y$ be two strings in $\Sigma^*$. $(x \neq y)$
Suppose there is a string $z \in \Sigma^*$ such that

$xz \in L$ but $yz \notin L$ or vice-versa,
ie $xz \notin L$ and $yz \in L$.

$\underbrace{\phantom{xz \notin L \text{ and } yz \in L.}}$

$xz \in L$ XOR $yz \in L$, ie. $L(xz) \neq L(yz)$

Then any possible DFA $M = (Q, \Sigma, \delta, s, F)$ s.t. $L(M) = L$ must process $x$ and $y$ from $s$ to different states $p, q$.

Write $\underline{x \equiv_L y}$ if for all $z \in \Sigma^*$, $L(xz) = L(yz)$

Then $x \not\equiv_L y$ means $(\exists z \in \Sigma^*)\ L(xz) \neq L(yz)$ as above

Note: For any $L$ and any $x, y \in \Sigma^*$

- $x \equiv_L x$
- $x \equiv_L y \leftrightarrow y \equiv_L x$ because the condition $L(xz) = L(yz)$ is symmetric.
- $w \equiv_L x \wedge x \equiv y \implies$ for all $z \in \Sigma^*$ $L(wz) = L(xz) \wedge L(xz) = L(yz)$ $\implies$ for all $z \in \Sigma^*$, $L(wz) = L(yz)$, ie. $\implies w \equiv_L y$.

Thus $\equiv_L$ is an equivalence relation on $\Sigma^*$.



It __partitions__ $\Sigma^*$ into __equivalence classes__.

There might be finitely or $\infty$-many equiv. classes.

$x \not\equiv_L y$ means $x$ and $y$ are in different classes

Myhill-Nerode Theorem (1958, independently! in depth). (John Myhill UB Math, Anil Nerode Cornell)

A language $L$ is regular $\Leftrightarrow$ the relation $\equiv_L$ has finitely many equiv. classes.

__Proof:__ $\Rightarrow$ (contraposed) states: if $\equiv_L$ has $\infty$-many equiv classes then $L$ is not regular.

By $L$ having $\infty$-many equiv classes, we can make an infinite set $S$ by __choosing__ __one__ string from each cla[ss]

$S$ has the property: $(\forall x, y \in S$ s.t. $x \neq y)$ $x \not\equiv_L y$.

Suppose there were a DFA $M$ s.t. $L(M) = L$. It would have some finite number $K$ of states. But $S$ has more than (any) $K$ strings, and so $M$ would be forced to process some pair $x, y \in S$ to the same state. Contradiction

Proof, Part 2 ($\Leftarrow$): $\equiv_L$ has finitely many equiv classes $\Rightarrow$ L is regular.

For every equivalence class $[x]_L$, we can think of a least string in the class as its "shortest name"

Define $M = (Q, \Sigma, \Delta, s, F)$ where
$Q = \{\text{equiv classes}\}$, $s = [\varepsilon]$, $F = \{[x] : x \in L\}$
and for all $c \in \Sigma$ and $[x] \in Q$ where $x$ is "the" name

$$\Delta(\underbrace{[x]}_{\text{"The" name}}, c) = \underbrace{[xc]}_{\text{Might have a lesser name but its the same class.}}$$

Then $L(M) = L$.

And if there are only finitely many equiv. classes, then M really is a DFA s.t. $L(M) = L$, so L is regular. ☒

Self-study: Convince yourself that for all $x, y, c$

$$[xc] = [yc] \Leftarrow [x] = [y]$$

This says that "$\Delta$ is well defined".