CSE596 Lecture Wed Sep 19 Fall 2018

For each $K \geq 1$, define (over $\Sigma = \{0,1\}$)

$L_K = \{x :$ the Kth bit from the right is a 1$\}$

Regexp: $\alpha_K = (0+1)^* \cdot 1 \cdot (0+1)^{\wedge (K-1)}$

This "powering" abbreviation makes α_K use only (over ASCII) $15 + \lceil \log_2 K \rceil$ characters

NFA: N_K =

K-1 steps

$K+1$ states

$\approx O(K)$ ASCII encoding

More Succinct ↑

Less Succinct, Verboser ↓

But, every DFA M_K s.t. $L(M_K) = L_K$ needs 2^K states!

Proof: Take $S = \{0,1\}^K$. **Claim**: S is PD for L_K, (pairwise distinguishable)

i.e. $(\forall x, y \in S, x \neq y)\ \underbrace{(\exists z \in \Sigma^*)\ L_K(xz) \neq L_K(yz)}_{x \not\sim_{L_K} y,\ \text{i.e. } [x] \neq [y]}$.

Since $\|S\| = 2^K$, proof follows.

Goal: show there exists a string $z \in \Sigma^*$ s.t. .....

Take $z = ..$

Let any $x, y \in S$, $x \neq y$ be given.

By $x \neq y$, there is a place j (numbering from 1 left to right) such that

x has a 0 in place j [1 ... | 0 | ...] 00000 Then $xz \notin L_K$

but y has a 1 in place j [... | 1 | ...] 000...00 $yz \in L_K$.

$K-j$ places. $j-1$

~~(or vice-versa)~~ Take $z = 0^{j-1}$

Either way (vice-versa too): $L_K(xz) \neq L_K(yz)$. Since $x, y \in S$ are an arbitrary pair, S is PD for L_K. $\boxtimes$

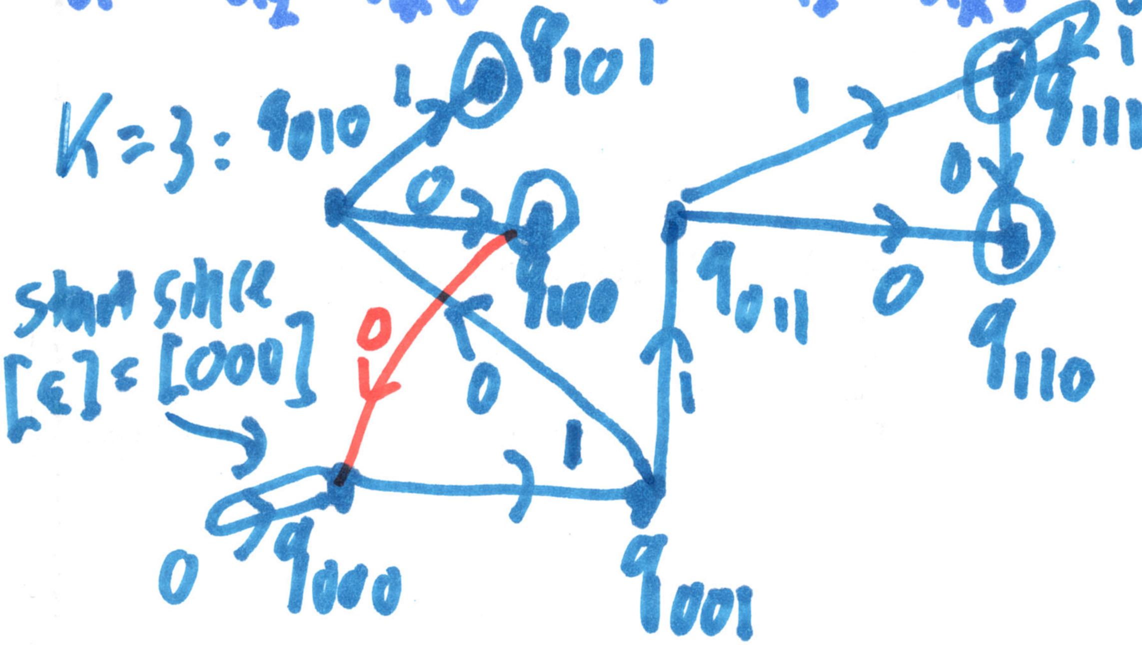
Furthermore, 2^K states suffice: Build M_K with each state q_u representing a string $u \in S$. The state q_K remembers the last K bits read. Transitions

$$\delta(q_u, 0) = q_{u'} \quad \delta(q_u, 1) = q_{u''}$$

$u' = u_2 \cdots u_K 0 \qquad u'' = u_2 \cdots u_K 1$

$F = \{q_u : u_1 = 1\}$

For $|u| < K$, consider $u' = 0^{K-|u|}u$.



So 8 states are necessary and sufficient for L_3, and 2^K for L_K.

Consider $L_\infty = \bigcup_{K=1}^{\infty} L_K$. How many equiv. classes?

$L_\infty = \{x \in \{0,1\}^* :$ for some $K \geq 1$, the Kth char from right in x is a 1.$\}$

$= \{x : x \text{ has a } 1\} = 0^* 1 (0+1)^*$.

DFA: 

2 states and $[0] \neq [1]$ so this is minimum

$L'_\infty = \{0^K x :$ the Kth char from right in x is a 1, $K \geq 1\}$.

now $y = 100$ is no longer in L'_∞, nor 0100.

How about $x = 00010$? yes: $y = \underbrace{00}_{k=2} \cdot \underbrace{010}_{\text{"}x\text{"}}$.

Prove: L'_∞ is not regular. Method:

<u>Take</u> $S = 0^+$. Clearly S is infinite; we will show S is <u>PD</u> for L'_∞.

Let any $x, y \in S$, $x \neq y$, be given. Then there are numbers $m, n \geq 1$ s.t. $x = 0^m$ and $y = 0^n$ and $m < n$

Without loss of generality, "x" can refer to the <u>shorter</u> one. "wlog."

<u>Take</u> $z = 10^{m-1}$ OK since $m \geq 1$ by $S = 0^+$.

Then $xz = 0^m 1 0^{m-1} \in L'_\infty$ since it has a 1 in place m from right

but $yz = 0^n 1 0^{m-1} \notin L'_\infty$? <u>no</u>

$= 0^m \cdot \underbrace{0^{n-m} 1 0^{m-1}}_{\text{still good}}$, i.e. still in L

<u>Back up</u>!

Take $z = 10^{n-1}$

Then $xz = 0^m 1 0^{n-1} \notin L'_\infty$ since m is too low

while $yz = 0^n 1 0^{n-1} \in L'_\infty$ "just right"

Thus $L'_\infty(xz) \neq L'_\infty(yz)$, and since $x, y \in S$ are arbitrary and S is infinite, L'_∞ is not regular by MNT