CSE 596 Lecture Mon Oct 8 Fall 2018

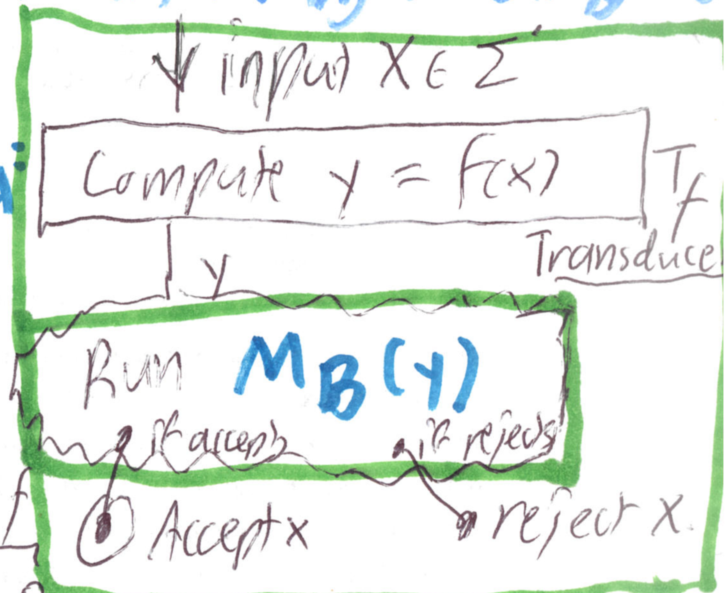
Reduction Meta-Theorem: Suppose B is ~~decidable~~ c.e. in poly time
(Incorporates that the reductions are transitive)

and $A \leq_m^p B$. Then A also is ~~decidable~~ c.e. in poly time

Proof: Take a ~~total~~ Turing $p(n)$-time machine M_B s.t. $L(M_B) = B$, for some polynomial $p(n) = O(n^k)$.

Build a machine M_A for A as follows, using a total $q(n) = O(n^\ell)$ time machine T_f computing the poly-time computable $\leq_m$ reduction f.

M_A:



Then for all $x \in \Sigma^*$, $x \in A \iff f(x) \in B$ (by def^n of $A \leq_m B$ via f)

We deduced this one $\Updownarrow$ $\Updownarrow$ hypothesis $L(M_B) = B$

M_A accepts $x \xLeftrightarrow[\text{code}]{} M_B$ accepts $y = f(x)$

Thus $L(M_A) = A$.

and M_A is total. Thus A is decidable.

$\therefore A \in P$.

How can we bound the running time of M_A? Not only could $T_f(x)$ take $O(|x|^\ell)$ time, it could output y of length $\approx n^\ell$. $|y| = n' \approx n^\ell$.

M_B runs in time $O(|y|^k) = O(n'^k) = O((n^\ell)^k) =$ time $O(n^{k\ell})$. Still a (bigger) polynomial time!

Corollary: If B is co-c.e. and $A \leq_m B$ then A is co-c.e.

Proof: By B being co-c.e., we can take a TM M_B st. $L(M_B) = \tilde{B}$. ($\tilde{B}$ is c.e.)

Theorem gives a TM M_A st $L(M_A) = \tilde{A}$ because

$A \leq_m B \equiv_{def}$ there is a function f st. for all $x \in \Sigma^*$, $x \in A \Leftrightarrow f(x) \in B$.

$\Updownarrow$

$\tilde{A} \leq_m \tilde{B} \equiv_{def}$ there is f st. $(\forall x)$ $x \in \tilde{A} \Leftrightarrow f(x) \in \tilde{B}$

$\Uparrow$ and $\tilde{B}$ is c.e.

M_A st. $L(M_A) = \tilde{A}$.

$\Downarrow$

A is co-c.e. $\boxtimes$

Applications come from the contrapositive:

Suppose $A \leq_m B$.

- If A is undecidable then B is undecidable.
- If A is not c.e. then so is B.
- If A is not co-c.e., then B is not either.

"The" Halting Problem:

INST: A TM M, and an input y to M

$HP_{TM} = \{\langle M, y\rangle : M(y)\downarrow\}$

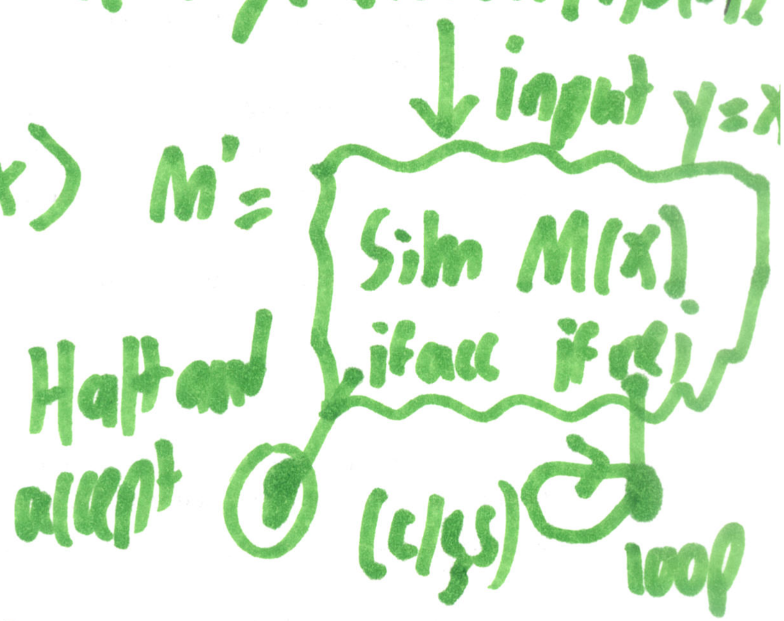
QUES: Does $M(y)\downarrow$? "halt"

Show $AP_{TM} \le_m HP_{TM}$, so HP_{TM} is undecidable, indeed not co-c.e. (it *is* c.e.)

Reduction function f needs to map $\langle M, x\rangle$ to $\langle M', y\rangle$ s.t. $M'(y)$ halts when and only when $M(x)$ accepts.

Do with $y = x$ (but more general on Friday)

$\langle M, x\rangle \hookrightarrow \langle M', x\rangle$ $M' =$



If $M(x)$ halts w/o accepting, $M'(x)$ doesn't halt and If $M(x)$ doesn't halt, $M'(x)$ doesn't halt either. So the traditional Halting Problem is undecidable, indeed its language is (c.e. but) not co-c.e.