CSE596 Lecture Fri 10/12 Fall 2018

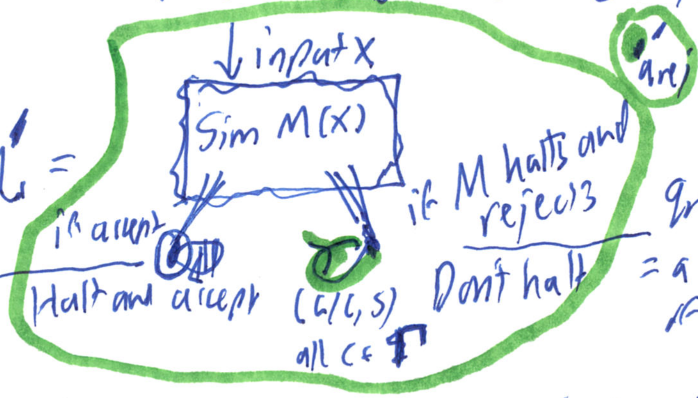
$ALL_{TM} = \{ \langle M \rangle : L(M) = \Sigma^* \}$.

$TOT = \{ \langle M \rangle : M$ is total, i.e. $\forall x\, M(x)\downarrow$

Show $ALL_{TM} \leq_m TOT$. We need to give a computable function $f : \underbrace{\{\text{codes of TMs}\}}_{\equiv \mathbb{N} \text{ via Gödel \#s}} \to \{\text{codes of TMs}\}$ such that for all codes $\langle M \rangle$, $f(\langle M \rangle)$ is the code of a TM M' such that $\langle M \rangle \in ALL_{TM} \iff \langle M' \rangle \in TOT$.

I.e. $\langle M \rangle \xrightarrow{f} M'$ st. $\overbrace{\forall x\, M'(x)\downarrow}^{f(M) \in TOT} \iff \overbrace{L(M) = \Sigma^*}^{M \in ALL_{TM}}$.

$\langle M \rangle \xrightarrow{f} M' =$



q_{acc} of M $= q_{acc}$ of M'

q_{rej} of M = a loop state of M'.

If we demand that M' has a q'_{rej} state it can be elsewhere and isolated

Major Points.

Construction

Computable: f is a simple task of adding a loop $(c/c,S)$ at q_{rej} for all $c \in \Gamma$. In fact f is streamable in linear time, so $ALL_{TM} \leq^p_m TOT$ too.

Correctness: For all arguments $\langle M \rangle$ to f:

$\langle M \rangle \in ALL_{TM} \equiv L(M) = \Sigma^* \Rightarrow (\forall x)\, M(x)$ accepts.

$\Rightarrow (\forall x)\, M'(x)\downarrow$ (and accepts too) $\equiv \langle M' \rangle \in TOT$.

But $\langle M \rangle \notin ALL_{TM} \equiv (\exists x)\, M(x)$ does not go to q_{acc}.

$\Rightarrow$ either $M(x)$ goes to the q_{rej} of $M \Rightarrow M'(x)\uparrow$, indeed, loops in q_{rej} of M

or $M(x)$ doesn't halt $\Rightarrow M'(x)\uparrow$ also because it otherwise simulates $M(x)$

$\therefore \langle M \rangle \notin ALL_{TM} \Rightarrow (\exists x)\, M'(x)\uparrow \equiv \neg(\forall x)\, M(x)\downarrow \equiv \langle M' \rangle \notin TOT$.

$\therefore \langle M \rangle \in ALL_{TM} \Rightarrow f(M) \in TOT$

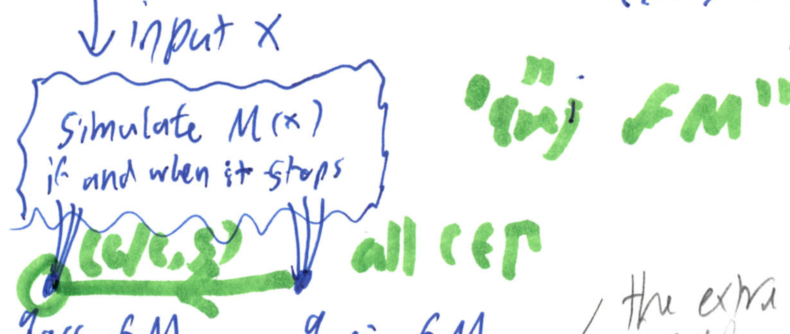
$\langle M \rangle \notin ALL_{TM} \Rightarrow f(M) \notin TOT$

$\therefore f$ is a correct mapping red^n from ALL_{TM} to TOT. ☒

Show TOT $\le_m$ ALL_{TM}. We need a function $g: \{\text{TM codes}\} \to \{\text{TM codes}\}$ s.t. for all M, M is total $\iff g(M) \in ALL_{TM}$

i.e. $\forall x\ M(x)\downarrow \iff L(M'') = \Sigma^*$ where $M'' = g(M)$ i.e. $\langle M'' \rangle = g(\langle M \rangle)$

Construction: Map

$\langle M \rangle \xrightarrow{g} M'' =$ 

Complexity: g is again a simple code edit ∴ linear time.

For correctness, we want $\forall \langle M \rangle$:

$\langle M \rangle \in TOT \equiv \forall x\ M(x)\downarrow \Rightarrow (\forall x)\ M''(x)$ goes either to q_{acc} or to q_{reject} of M $\Rightarrow \forall x\ M''(x)$ accepts $\equiv M'' \in ALL_{TM}$

$\langle M \rangle \notin TOT \equiv \exists x\ M(x)\uparrow \Rightarrow (\exists x)\ M''(x)\uparrow$ too $\Rightarrow \exists x\ M''$ does not accept x $\equiv M'' \notin ALL_{TM}$

the extra arc implies this.

So we got $ALL_{TM} \le_m TOT$ and $TOT \le_m ALL_{TM}$: we write $ALL_{TM} \equiv_m TOT$

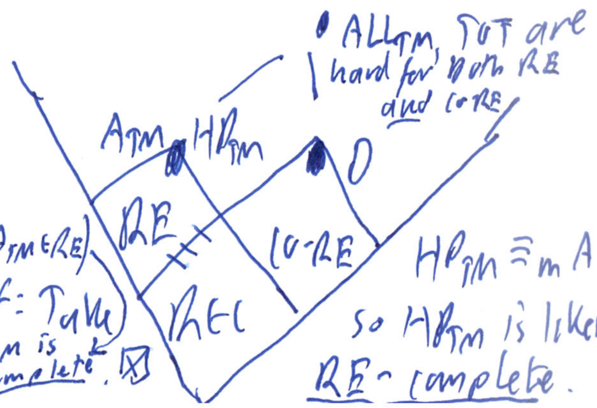
In fact we got $ALL \le_m^p TOT$ & $TOT \le_m^p ALL_{TM}$ so $ALL_{TM} \equiv_m^p TOT$

Similarly $A_{TM} \le_m^p HP_{TM}$ & $HP_{TM} \le_m^p A_{TM}$ so $A_{TM} \equiv_m^p HP_{TM}$.

These are equivalence relations, since $\le_m$ and $\le_m^p$ are reflexive & transitive

Defn: A language B is hard for a class $\mathcal{C}$ of languages under a reducibility $\le_r$ (so far $\le_r = \le_m$ or $\le_m^p$) if for all $A \in \mathcal{C}$, $A \le_r B$.

If also $B \in \mathcal{C}$, then B is complete for $\mathcal{C}$ (under the reducibility)



Thm: For all $A \in RE$, $A \le_m^p A_{TM}$. Quick proof: Take M_A s.t. $L(M_A) = A$ and do $f(x) = \langle M_A, x \rangle$. (noting $A_{TM} \in RE$) So A_{TM} is RE-complete. ☒

$HP_{TM} \equiv_m A_{TM}$ so HP_{TM} is likewise RE-complete.