CSE 596 Lecture Wed 10/17 Fall 201

Last time: $A_{TM} \leq_m ALL_{TM}$, so ALL_{TM} is not co-c.e.

Show $D_{TM} \leq_m ALL_{TM}$, so ALL_{TM} is not c.e.

Trick 3: "Delay Switch".

Need $\langle M \rangle \xrightarrow{g} M'$ s.t.

$\langle M \rangle \in D_{TM} \equiv M$ does not accept its own code $\Rightarrow L(M') = \Sigma^*$

$\langle M \rangle \notin D_{TM} \equiv \langle M \rangle$ does accept $\langle M \rangle \Rightarrow L(M') \neq \Sigma^*$

In the latter case we will make $L(M')$ to be finite.

Construction:

Given any TM M, map $M \xrightarrow{g} M' =$

↓ input x

Take $n = |x|$
Simulate $M\langle M \rangle$ for up to n steps only.

if it accepted: Reject x

if it did not accept $\langle M \rangle$: Accept x

i.e. it rejected or it is still going.

Key "delay" point: If M does accept $\langle M \rangle$ it does so in some number t of steps, so that whenever $|x| = n \geq t$, this acceptance is discovered and x is rejected by M'.

So $\langle M \rangle \notin D_{TM} \Rightarrow (\exists t)\ L(M') \subseteq \{0,1\}^{<t} \Rightarrow L(M')$ is finite $\Rightarrow L(M') \neq \Sigma^*$.

How about: INST: A TM M'

Name QUES: Is $L(M')$ infinite?

I_{INF}, the "Index Set" of the class of infinite c.e. languages

- Above shows $D_{TM} \leq_m I_{INF}$.
- $A_{TM} \leq_m I_{INF}$ by the "All or Nothing Switch".

Show $I_{INF} \leq_m ALL$

Def^n: For any subclass $\mathcal{C}$ of the r.e. languages, its index set is $I_{\mathcal{C}} =_{def} \{\langle M \rangle : L(M) \in \mathcal{C}\}$.

As a problem, it is: INST: A TM M
Ques: Is $L(M)$ in $\mathcal{C}$?

The answer depends only on the input/output behavior of M, not on how it is coded. "extensional"

Examples:

E_{TM} is the index set $I_{\{L : L = \emptyset\}} = I_{\{\emptyset\}}$.

$NE_{TM} = \{\langle M \rangle : L(M) \neq \emptyset\} = I_{\{L : L \neq \emptyset\}} = \Sigma^* \setminus E_{TM}$ if all strings are valid codes

$ALL_{TM} = I_{\{\Sigma^*\}}$. $I_{INF} = \{\langle M \rangle : L(M) \text{ is infinite}\}$

$I_{REG} = \{\langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular}\}$ which includes DTMs that are not DFAs.

$I_P = \{\langle M \rangle : L(M) \in P\}$ irrespective of whether M itself runs in polynomial time.

What is $I_\emptyset$, the index set of the empty class?

What is I_{RE}, the index set of the class of all r.e. languages?

$I_{RE} = \mathbb{N}$ for Gödel #s $\equiv \Sigma^*$. So $I_\emptyset$ is the complement of Σ^*, i.e. $I_\emptyset = \emptyset$.

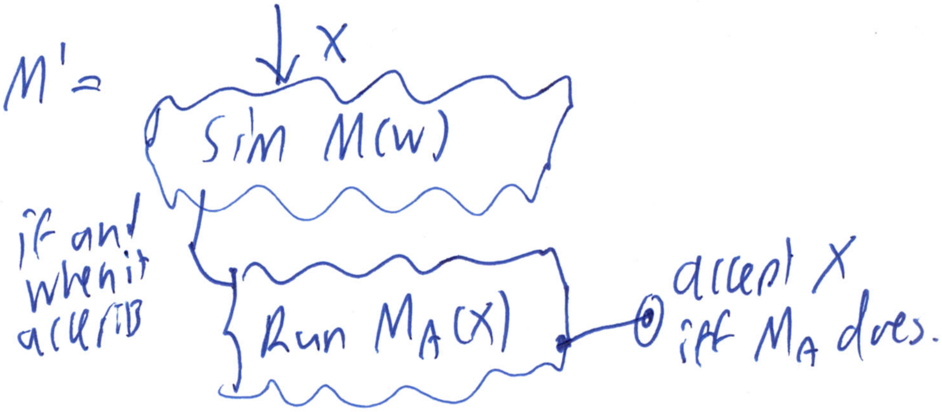
Rice's Theorem: Those are the only two decidable index sets.

"Moral" (in the version extended to classes of functions) Every nontrivial extensional property of programs is undecidable.

Proof: Given $I_\mathcal{C}$ where $\mathcal{C} \neq \emptyset$ and $\mathcal{C} \neq RE$, so there is a c.e. language A such that either: (a) $\emptyset \in \mathcal{C}$, $A \notin \mathcal{C}$ (*)

Take an M_A s.t. $L(M_A) = A$. or (b) $A \in \mathcal{C}$, $\emptyset \notin \mathcal{C}$.

Build $\langle M, w\rangle \xrightarrow{f} M' =$



In case (a),

$\langle M,w\rangle \in A_{TM} \Rightarrow L(M') = A \Rightarrow M' \notin I_\mathcal{C}$

$\langle M,w\rangle \notin A_{TM} \Rightarrow L(M') = \emptyset \Rightarrow M' \in I_\mathcal{C}$ $\quad \therefore A_{TM} \leq_m \widetilde{I_\mathcal{C}}$.

In case (b) $\quad \therefore A_{TM} \leq_m I_\mathcal{C}$

$\langle M,w\rangle \in A_{TM} \Rightarrow L(M') = A \Rightarrow L(M') \in \mathcal{C} \Rightarrow M' \in I_\mathcal{C}$

$\notin A_{TM} \Rightarrow L(M') = \emptyset \Rightarrow M' \notin I_\mathcal{C}$ since $\emptyset \notin \mathcal{C}$ in this case

Either way, $I_\mathcal{C}$ is undecidable. ☒

Example: $\mathcal{C} = REG$, case (b) applies with $A = \{\text{palindromes}\}$, so I_{REG} is undecidable.

Added:

(*) I could have worded this as, "Given any class $\mathcal{C}$, first suppose (a) that the empty language $\emptyset$ is in $\mathcal{C}$. E.g., $\mathcal{C}$ is the class of regular languages. Then by $\mathcal{C} \neq RE$, there is some c.e. language A that is not in $\mathcal{C}$. Take an M_A s.t. $L(M_A) = A$, eg if A = the set of palindromes. If (b) $\emptyset$ is <u>not</u> in $\mathcal{C}$, then by $\mathcal{C} \neq RE$ we can take some A that <u>is</u> in $\mathcal{C}$, take M_A s.t. $L(M_A) = A$, and continue as above.