CSE 596 Lecture Mon Oct 29 Fall 2018

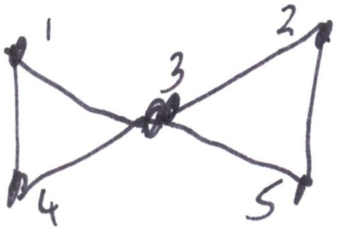
NP-completeness via $\leq_m^p$ reductions from (3)SAT. Examp

INDEPENDENT SET $G = (V, E)$

INST: An undirected graph G, and a number $K \leq n \equiv |V$

QUES: Is there a set $S \subseteq V$, $|S| \geq K$ such that no two nodes in S are adjacent? [**Extra**: If so, output optimal S and K.

Example Graph G=



$K = 3$ answer is no. $S = \{1,2\}, \{1,5\}$
$K = 2$ answer is yes: $\{4,2\}$ or $\{4,5\}$
$K = 1$: yes but not optimal [don't use node 3:

Theorem: INDSET is NP-complete: In NP and 3SAT $\leq_m^p$ INDSE

Proof: "In NP": Given G, if $\langle G, K\rangle \in L_{INDSET}$ then we can guess $S \subseteq$

$S \subseteq V$ means $\|S\| \leq n = |V|$ and lengthwise, $|S| \approx n \log n$.

Verify: Check $(\forall u, v \in S, u \neq v)\ (u,v) \notin E$. $\therefore$ poly in $n \leq |G|$.

How many pairs to check? $\binom{|S|}{2} \leq \binom{n}{2} = O(n^2)$, polynomial in n. $\therefore$ INDSET $\in$ NP

$L_{INDSET} = \{\langle G, K\rangle : (\exists S : |S| \leq n)[(\forall u, v \in S, u \neq v)\ (u,v) \notin E]\}$.

This is $(\exists^{poly})$[poly-time] form. Hence in NP.

Why can't we try all S-es? There are 2^n subsets $S \subseteq V$.
Hence trying them all leads to exponential time. $K \approx \frac{n}{2}$ very typical

3SAT $\leq_m^p$ INDSET. Let any $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ be given

$\phi \xrightarrow{f} G_\phi, K_\phi$ m = # clauses, each with (up to) 3 variables, n = # va

eg $\phi(x_1, \ldots, x_n) = (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x_3)$.

CCC outline: **Construction**: Show how to build G_ϕ and define K_ϕ from ϕ. ②

Complexity: Say why the building is easy [often "streamable"

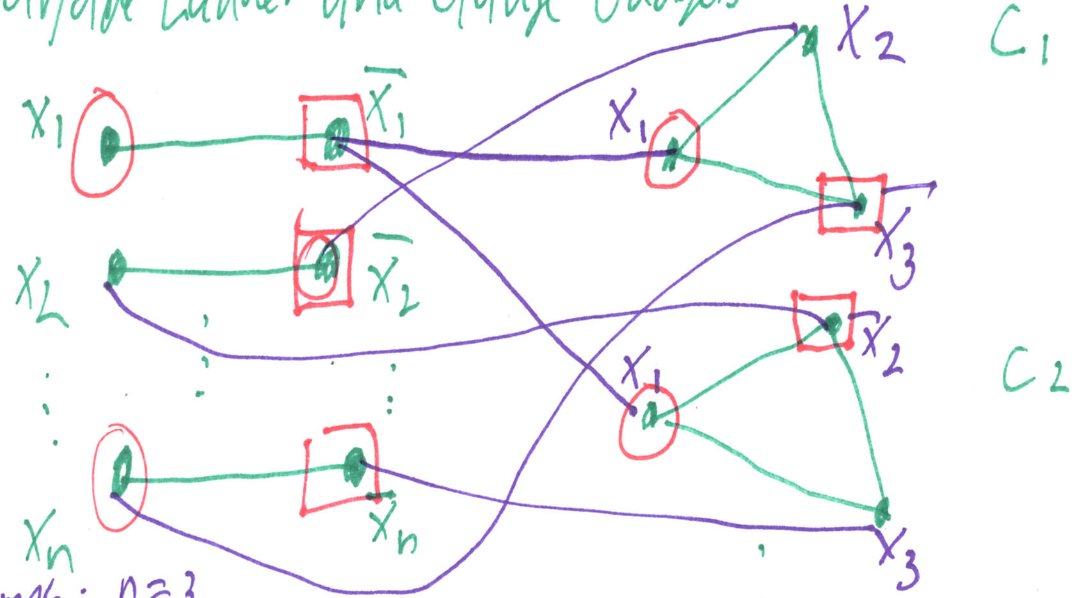
Correctness: Show that [example: $\phi (x_1 \vee x_2 \vee \bar{x}_3) \wedge (x_1 \vee \bar{x}_2 \vee x$

ϕ satisfiable $\Rightarrow$ G has an~~d~~ ind-set S of size K_ϕ

ϕ not satisfiable $\Rightarrow$ all ind sets in G have size $< K_\phi$.

Idea Set up a correspondence (not necessarily 1-1) between sat assgts $(a_1, \ldots, a_n)$ to ϕ and solutions S to G_ϕ for a critical value of K_ϕ.

"Variable Ladder and Clause Gadgets"



We have $|V|$ = der N equal to $2n + 3m$.

Max possible size of S is $n + m$

m clauses

$= K_\phi$

example: $n = 3$

Key Idea: Put in "crossing edges" from Clause Gadgets to rungs so that S can have size K_ϕ (one from each rung and triangle only if the choices of rung variables induce a sat. assgt.

Eg. ▣ choices induce the assignment (0,0,0) which satisfies ϕ via $\bar{x}_3$ in C_1, $\bar{x}_2$ in C_2.

◉ choices include $x_1 = 1$, whereupon $x_2 = 0$, $x_3 = 1$ don't matter: ϕ satisfied via x_1 in both clauses

Blue lines are the "crossing edges" which depend on details of the particular ϕ.

The green "ladder edges" and "gadget edges" only depend on n and m from ϕ. They define $K_\phi = n + m$ regardless of details of ϕ as the goal value. Next time: formal details...