CSE596 Lecture Wed Nov 7 Fall 2018

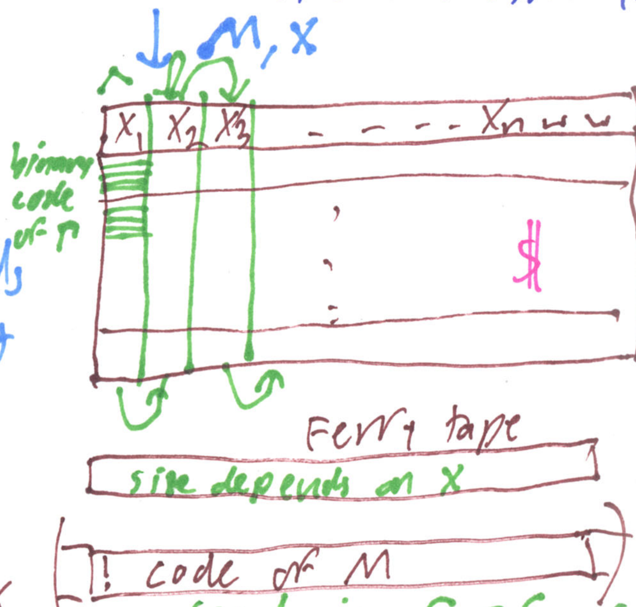
Time and Space Hierarchy Theorems.

Based on Efficient Universal Simulation:

TM M with
- any # k of tapes
- any size alphabet Γ
- any input x
- Given M is total: own time $t_1(n)$, own space $s_1(n)$

Not mapped to a machine M' that depends on $\langle M, x\rangle$ but input to one machine M_{U_3}

3 tapes, binary alph. plus blank



Tape 1 has $k+1$ tracks using alph $\{0,1,\sqcup\}$ so they are braided

fixed size $C = C_M$ for any M

Theorem (4 in notes) $M_U(\langle M,x\rangle)$ can operate within: own space $\leq C_M \cdot s_1(n) + C_M$ / own time $\leq C_M \cdot t_1(n)\log t_1(n)$ for some const. C_M that depends on M, but not on x (or $n = |x|$)

Space Hierarchy Theorem:

If $s_1(n) = o(s_2(n))$, and if $s_2(n)$ is "reasonable", then $DSPACE[s_1(n)] \subsetneq DSPACE[s_2(n)]$

Time Hierarchy Theorem:

If $t_1(n)\log t_1(n) = o(t_2(n))$ (and $t_2(n)$ is reasonable/"clockable") then $DTIME[t_1(n)] \subsetneq DTIME[t_2(n)]$

$t(n) = o(u(n))$ means $\lim_{n\to\infty} \frac{t(n)}{u(n)} \to 0 \Rightarrow (\forall C > 0)(\exists n_0)[\forall n \geq n_0]\ C \cdot t(n) \leq u(n)$.

Example: $t(n) = n\log n$ i.e. $t'(n) = \log_2 n$
$u(n) = n^{1.00001}$ $u'(n) = n^{0.00001}$ can prove via L'Hopital's Rule

Simpler example: $n = o(n\log^2 n) = o(n^2) = o(n^3) = o(\text{poly in general}) = o(2^n)$

$\therefore DLIN \subset DTIME[n\log^2 n] \subset DTIME[n^2] \subset DTIME[n^3] \subset P \subset EXP$.

Proof uses a special Diagonalizing Machine M_s or M_t:

M_s:

$\downarrow x$

$\boxed{\text{Decode } x =: \langle M \rangle y}$ if not possible → reject

compute $t_2(n)$ $\boxed{\text{Lay out } s_2(n) \text{ cells to use for } M_{U_3}}$

Go to start state of $M_{U_3}(M, x)$
$\equiv$ run M on $\langle M \rangle y$

If run oversteps the bound, reject.

If the run completes and M accepts x, then M_s rejects x
or if M rejects x, M_s accepts x.

M_s always runs within $s_2(n)$ tape cells of its own.

$\therefore L(M_s) \in DSPACE[s_2(n)]$.

Suppose $L(M_s) \in DSPACE[s_1(n)]$. Then there would exist a DTM Q that accepts $L(M_s)$ in space $s_1(n)$. Take the constant C_Q from Theorem 4.

By $s_1(n) = o(s_2(n))$, there is an n_0 s.t. $\forall n \geq n_0$, $C_Q \cdot s_1(n) \leq s_2(n)$.
$C_Q \cdot s_1(n) + C_Q \leq s_2(n)$

Consider any input $x = \langle Q \rangle y$, $|y| = n - |Q|$

$M_s(x)$ runs $M_{U_3}(Q, x)$ and it stays within $s_2(|x|) = s_2(n)$ cells.
So the run completes but it gives the opposite answer:
If Q accepts x, $M_s(x)$ rejects. If Q rejects x, $M_s(x)$ accepts.
$\therefore M_s(x) \neq Q(x)$, which contradicts $L(M_s) = L(Q)$.
$\therefore Q$ cannot exist, so $L(M_s) \notin DSPACE[s_1(n)]$. ☒

The case for M_t and time is similar where M_t has a separate clock that counts up to (or down from) $t_2(n)$. We get n_0 s.t. for $n \geq n_0$, (by $n + t_1(n) \log t_1(n) = o(t_2(n))$)

we get $\boxed{\underbrace{n+1}_{\text{time to decode } x} \underbrace{+ t_1(n)}_{\text{time to start clock}} + C_Q \cdot t_1(n) \log t_1(n) \leq t_2(n).}$

So the run $M_t(x)$ with $|y| = n - |\langle Q \rangle|$, $x = \langle Q \rangle y$ finishes before $t_2(|x|)$ steps are up, so we get the same contradiction. ☒