CSE596 Lecture Mon 11/12 Fall 2018

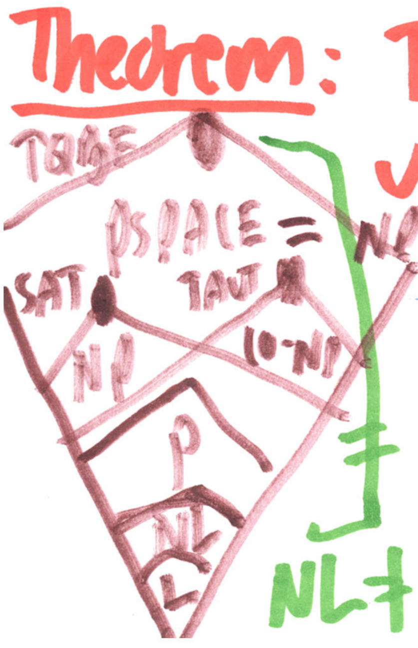
Def^n: A quantified Boolean formula (<u>qbf</u>) adds quantifiers $\exists x_i$ $\forall x_j$ to an ordinary B.F.

<u>note</u>: $\phi(x_1, \dots, x_n)$ is satisfiable iff the qbf $(\exists x_1) \cdots (\exists x_n)\, \phi(x_1 \dots x_n)$ is true.

$\psi(x_1, \dots, x_n)$ is a tautology iff the qbf $(\forall x_1) \cdots (\forall x_n)\, \psi(x_1 \dots x_n)$ is true.

Def^n: TQBF (aka QBF) is the language of all <u>true</u> qbf's. These can have alternating quantifiers
$(\forall x_1 \dots x_m)(\exists y_1 \dots y_r)(\forall z_1 \dots z_l)$ etc.

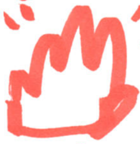
Theorem: TQBF is complete for PSPACE under $\leq_m^{\log}$ reductions



The proof will also show:

<u>Savitch's Theorem</u> For any "reasonable" $s(n) \geq \log_2 n$, NSPACE$[s(n)]$ is in DSPACE$[s(n)^2]$.

NL ≠ PSPACE



Proof: TQBF $\in$ DSPACE$[O(N)]$ because we can do brute-force evaluation of the qbf using only the space occupied by all the variables (hardware)

For completeness, let any $A \in$ DSPACE$[s(n)]$ be given, [where $s(n) = n^k$ for some k] Ah, I claim NSPACE$[s(n)]$

Take an NTM N s.t. $L(N) = A$ and N runs in space $s(n)$. To show $A \leq_m^{\log}$ TQBF we need to map

$x \xrightarrow{f} \psi_x$ s.t. $x \in A \iff \psi_x$ is true.

So we need to describe $f(x)$, given any x, and argue that f is computable in logspace.

Key idea: Think again of G_x, the ID graph with edges (I, J) s.t. $I \vdash_N J$. Then $x \in A \iff G_x$ has a path of length $2^{O(s(n))}$ "$\|G_x\|$" from the start ID $I_0(x)$ to $\left\{\begin{matrix}\text{an}\\ \text{the}\end{matrix}\right.$ accepting ID I_f.

$n = |x|$

Put $2^r =$ "$2^{O(s(n))}$", so $r = r(n)$.

For $0 \le j \le r$, define $\Phi_j(I, K)$ to hold iff
$I \vdash_N^* K$ in at most 2^j steps

So: $x \in A \Leftrightarrow \Phi_r(I_0(x), I_f)$.

$\Leftrightarrow (\exists J)\ \Phi_{r-1}(I_0(x), J) \wedge \Phi_{r-1}(J, I_f)$

Generally,

$$\Phi_j(I, K) \Leftrightarrow (\exists J)\ \Phi_{j-1}(I, J) \wedge \Phi_{j-1}(J, K).$$

$$\Leftrightarrow (\exists J)(\forall I', J'):$$

$$\left[\begin{array}{l} (I' = I \wedge J' = J) \\ \vee\ (I' = J \wedge J' = K) \end{array}\right] \Rightarrow \Phi_{j-1}(I', J').$$

This is a single branch recursion. At bottom

$$\Phi_0(I, K) =_{def} I = K \vee I \vdash_N^1 K.$$

Total size is $r \times |\Phi_0(I, K)| = O(r^2)$

Easy to compute and translate to qbfs $\rightarrow$ ved.