CSE596 Lecture Mon 11/19 Fall 2018

How Does Nature Compute?

$\Downarrow$

How Does Nature Concatenate?

<u>Strings</u>: Trivial operation: $a \cdot b = ab$
$x \cdot y = xy$
$A \cdot B = \{x \cdot y : x \in A \wedge y \in B\}$

Nature uses (linear) <u>operators</u> that we represent as matrices (over a standard basis).

Nature's concatenation of ~~matrices~~ operators A, B is <u>not</u> their product <u>$A \cdot B$</u>

All matrices we care about will be <u>square</u> $n \times n$ or $N \times N$ where n or N are <u>finite</u>.

This is <u>composition</u>

Convention: N usually stands for 2^n.

$$(A \cdot B)(u) = A(v) \text{ where } v = B(u).$$

⚠ Matrix product is not commutative.

Main Point: Matrix multiplication is highly "Lossy."

Instead, it is <u>Tensor Product</u>: $A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & & & \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$

How to visualize it: Say $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$:

As a matrix, if A is $n \times n$ and B is $r \times r$, then $A \otimes B$ is $(nr) \times (nr)$. If we form $\underbrace{A \otimes A \otimes \cdots \otimes A}_{k \text{ items}}$, then the matrix $A^{\otimes k}$ has size n^k. If $k \approx n$, this is exponential size

Example Hadamard Matrix (without normalizing) ②

$$H = H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \left(\text{normalized: } \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\right).$$

$$H \otimes H = \begin{bmatrix} 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} & -1\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ 00 \\ 01 \\ 10 \\ 11 \end{array} \quad 4 \times 4$$

$H \otimes H \otimes H$ is $2^3 \times 2^3$
ie 8×8

$H^{\otimes n}$ is $N \times N$ where $N = 2^n$.

Rule: $+1$ if $\langle \text{row} \cdot \text{col} \rangle = 0$
Multiply $\text{row}_i \cdot \text{col}_i$ for all i then add up mod 2.

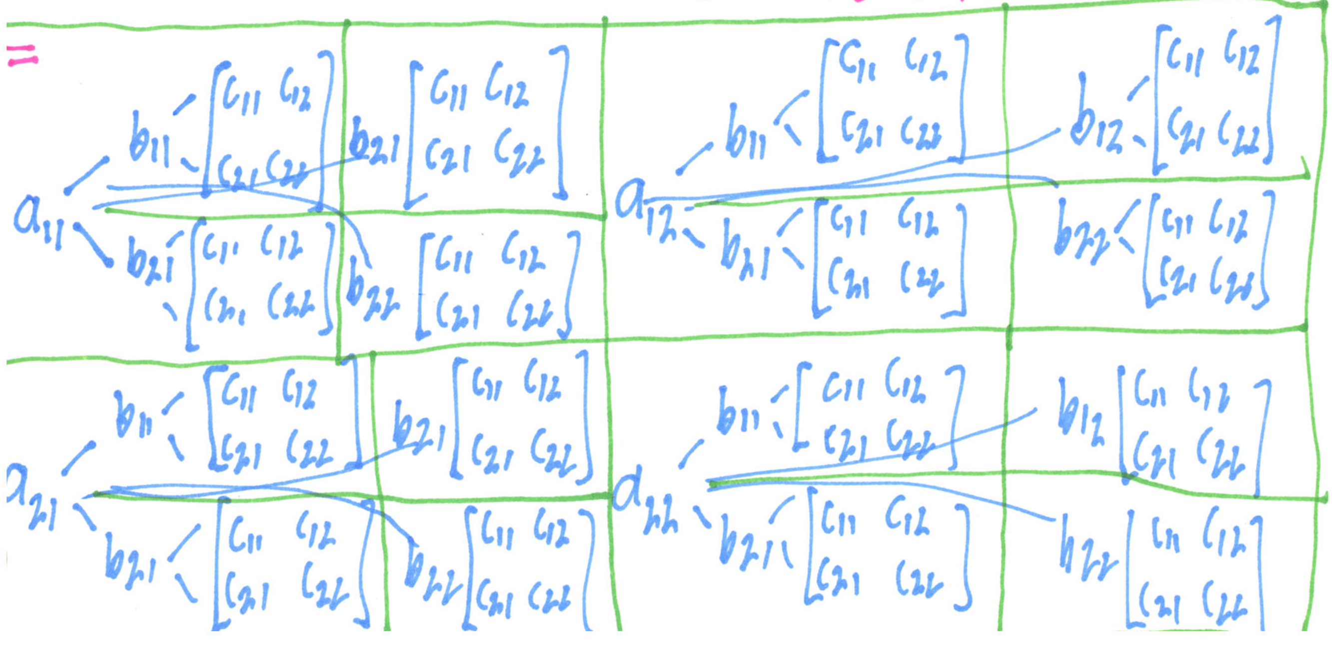
This rule defines $H^{\otimes n}[x, y]$ for any n.

[ADDED]
Why is this concatenation?
Consider $A \otimes B \otimes C$ where

$$A = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array} \quad B = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array} \quad C = \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \end{array} \begin{array}{c} \\ 0 \\ 1 \end{array}$$

The resulting 8×8 matrix — call it D — gives
$D[x_1 x_2 x_3, y_1 y_2 y_3] = A[x_1, y_1] \cdot B[x_2, y_2] \cdot C[x_3, y_3]$

=



for all binary strings $x, y \in \{0,1\}^3$

as you can check by labeling the eight coordinates 000, 001, ... 110, 111.