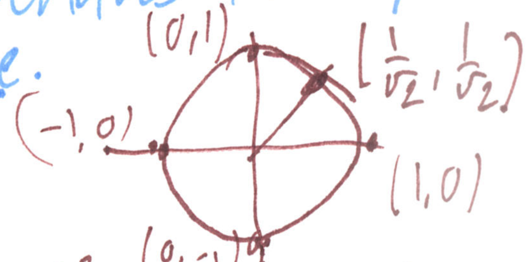
A Qubit is a physical system that behaves like a point on the unit complex circle.



Point (a,b) where $|a|^2+|b|^2=1$.

Called the pure state of the qubit.

Can be real. The entries a, b are complex amplitudes ≈ "Square Roots of Probabilities".

in $\mathbb{R}^2$: $x^2+y^2=1$

in $\mathbb{C}^2$: $|x|^2+|y|^2=1$

where if $x=s+it$

then $|x|^2=s^2+t^2$.

Components of any vector $(a_1, a_2, \dots, a_N) = \vec{v}$ correspond to multiples of basis vectors $e_1, e_2, \dots, e_N$

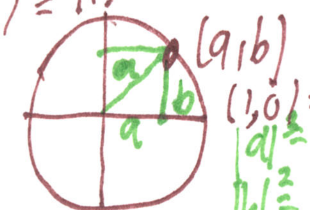
$\vec{v} = a_1\vec{e}_1 + a_2\vec{e}_2 + \dots + a_n\vec{e}_N$. By the same idea:

$(a,b) = \underbrace{a\cdot\vec{e}_1}_{\text{for the string } 0} + \underbrace{b\cdot\vec{e}_2}_{\text{for the string } 1}$

Outcome space for 1 qubit $= \{0,1\}^1 = \{0,1\} = |0\rangle, |1\rangle$

$(0,1) = |1\rangle$ $\quad |0\rangle = e_1, \quad |1\rangle = e_2$

So $(a,b) = \underbrace{a}_{\text{amplitude of outcome "0" when the qubit is measured}}|0\rangle + \underbrace{b}_{\text{amplitude of outcome "1".}}|1\rangle$



$|a|^2 = Pr[\text{measure} = 0]$

$|b|^2 = Pr[\text{measure} = 1]$

IMO the real "Axiom of Qubits" is not the {Pythagorean / Max Born} rule in isolation but the fact of its working under transformations of qubits.

For single qubits $\begin{pmatrix} a \\ b \end{pmatrix}$, the allowed transformations are given in the std. basis by 2×2 matrices over $\mathbb{C}$ that are unitary.

$U = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} au_{11} + bu_{12} \\ au_{21} + bu_{22} \end{pmatrix}$

A matrix U is <u>unitary</u> if $UU^* = U^*U = I$ $U^* = \overline{U}^T$.

U^t = transpose (or U^T) $\overline{U}$ complex conjugate of each entry.

Certainly implies that U is invertible, $U^* = U^{-1}$, so this preserves the structure of vectors

<u>Examples</u>. Hadamard Matrix $H = \begin{bmatrix}1&1\\1&-1\end{bmatrix}\cdot\frac{1}{\sqrt{2}}$

$HH^* = H^*H = HH = \begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}1&1\\1&-1\end{bmatrix} = \begin{bmatrix}2&0\\0&2\end{bmatrix}\cdot\frac{1}{2} = I$. $H^* = H$ since H is real and symmetric.

$X = \begin{bmatrix}0&1\\1&0\end{bmatrix}$ again $X^* = X$ $XX^* = X^2 = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix} = I$ NOT gate, really a prob^y switcher

$X\begin{pmatrix}a\\b\end{pmatrix} = \begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix}$

$Y = \begin{bmatrix}0&i\\-i&0\end{bmatrix}$ $Y^T = \begin{bmatrix}0&-i\\i&0\end{bmatrix}$ but $Y^* = \overline{Y}^T = \begin{bmatrix}0&i\\-i&0\end{bmatrix}$ again.

Some sources say $Y = \begin{bmatrix}0&-i\\i&0\end{bmatrix}$ instead. Doesn't really matter.

$YY^* = \begin{bmatrix}0&i\\-i&0\end{bmatrix}\begin{bmatrix}0&i\\-i&0\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$.

$Z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$ $Z^* = Z$ and $Z^2 = I$.   These four matrices, together with I, are named for <u>Wolfgang Pauli</u>. Nobel for Quantum theory

<u>Others</u>

$S = \begin{bmatrix}1&0\\0&i\end{bmatrix}$ $S^* = \begin{bmatrix}1&0\\0&-i\end{bmatrix}$ $SS^* = I$.

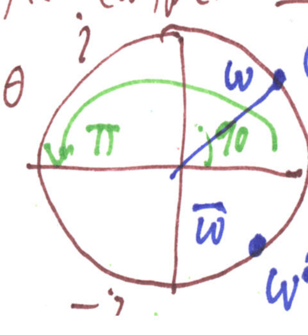
$T = \begin{bmatrix}1&0\\0&\sqrt{i}\end{bmatrix}$

$T^* = \begin{bmatrix}1&0\\0&\overline{\omega}\end{bmatrix}$, $TT^* = I$.

The planar circle of a single complex <u>scalar</u>

Multiplication = adding angles. θ

Points are $e^{i\theta}$. $e^{i(90°)} = e^{i\frac{\pi}{2}} = i$ $e^{\pi i} = -1$



$e^{\pi i/4} = s + it$ $s, t = 1/\sqrt{2}$