Multiple / Two-Qubit Systems: n qubits, $N = 2^n$ components; 2 qubits, 4 components.

Represent by unit vectors v in $\mathbb{C}^N$. {Pure states. Unit means $\|v\|^2 = \sum_{i=1}^{N} |v_i|^2 = 1$

Key Defn: v is *separable* if there are (wlog unit) vectors u, w s.t. $v = u \otimes w$.
Otherwise, v is *entangled*: Some examples ($N=4$, $n=2$)

$$j = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

We also call this $|+\rangle^{\otimes 2} = |+\rangle \otimes |+\rangle$

$\langle \cdot | =$ bra ↑ ↑ Ket

$$e_{00} = \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array}\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} = \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_0} \otimes \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{e_0} = \begin{pmatrix} 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = e_0 \otimes e_0$$

$$e_1 \otimes e_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_{11}.$$

$$e_0 \otimes e_1 = \begin{pmatrix} 1 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ 0 \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} = e_{01} \qquad e_1 \otimes e_0 = \begin{pmatrix} 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = e_{10}$$

All four std basis states. All separable.

In Dirac notation they are called $|00\rangle = |0\rangle \otimes |0\rangle$, $|01\rangle$, $|10\rangle$, and $|11\rangle$.

How about $\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$? $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so separable. Also called $|+\rangle \otimes |1\rangle$.

Now how about $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}} = \dfrac{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}{\sqrt{2}} = |0\rangle \otimes |+\rangle$ Notice: Ket-0 is not 0.

Ket = column vector, bra = conjugated row vector.

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} ? \overset{?}{=} \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}.$$

says $a = 0 \lor d = 0$
says $b = 0 \lor c = 0$.

Force either top or bottom entry to be 0 too. Hence impossible.

Hence this two-qubit state $= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is entangled. Likewise $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}$ = EPR pair $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. Either one is called a EPR Bell pair.

Multi-Qubit operators are (represented by) $N \times N$ matrices.

$$\text{CNOT} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix} = \begin{pmatrix}a\\b\\d\\c\end{pmatrix} \qquad \text{CNOT}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \text{CNOT}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

control

X

aka CX

Permutation matrix, deterministic two-qubit gate.

ie $[CX]\, e_{00} = e_{00} \quad [CX]\, e_{01} = e_{01}$

$$[CX]\, e_{10} = CX\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} = e_{11}$$

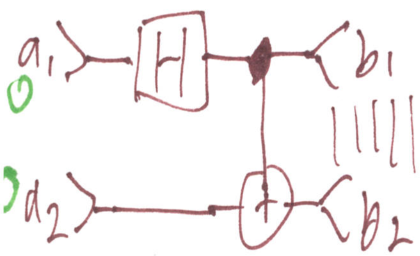
Similarly, $[CX]\, e_{11} = e_{10}$.

On basis states, if the first line has 1 then second line gets flipped. Else identity. Controlled-Bitflip.

We can also get them by tensor products.

$$\equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & 1\begin{bmatrix}1&0\\0&1\end{bmatrix} \\ 1\begin{bmatrix}1&0\\0&1\end{bmatrix} & -1\begin{bmatrix}1&0\\0&1\end{bmatrix}\end{bmatrix}$$

First Nontrivial Quantum Circuit: C_2



Matrix Form:

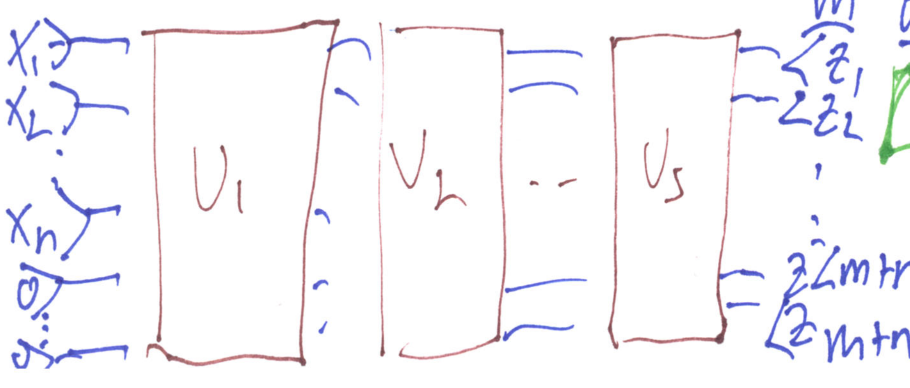
$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}\frac{\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&-1&0\\0&1&0&-1\end{bmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix}1&0&1&0\\0&1&0&1\\0&1&0&-1\\1&0&-1&0\end{bmatrix}}{\sqrt{2}}\overset{e_{00}}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

From the basis state $|00\rangle$, it produced the entangled EPR pair.

$$C_2 \cdot |10\rangle = \begin{pmatrix}1\\0\\0\\-1\end{pmatrix} = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$ which is likewise entangled

General Quantum Circuits have: n input qubits

m ancilla qubits initialized to $|0\rangle$.



At the end, some number $r \le m+n$ of the qubits are measured to give a final output $z \in \{0,1\}^r$.