Lectures and Reading. Next week will continue reductions showing NP-hardness from the ALR notes, ch. 28, section 4, and parallel coverage in DeBray’s notes.

(1) Show that the problem of deciding whether an undirected graph is connected belongs to polynomial time. Sketch an algorithm in pseudocode and estimate its asymptotic running time in terms of the number $n$ of vertices and number $m$ of edges. (18 pts.)

(2) A symmetric $n \times n$ 0-1 matrix $A$ represents an $n$-vertex undirected graph $G_A = (V,E)$ with $A_{ij} = 1 \iff G$ has an edge between vertex $i$ and vertex $j$. We’ll assume that $G_A$ has no self-loops; i.e., $A_{ii} = 0$ for all $i$. Represent $A$ in row-major order as a binary string of length $N = n^2$. Define $L_2$ to be the language of $A$ such that $V$ can be broken into $V_1 \cup V_2$ such that all edges in $G_A$ go between $V_1$ and $V_2$.

(a) Show that $L_2$ belongs to P. You don’t have to design a Turing machine $M$ to accept $L_2$—you can write pseudocode using data structures (such as sets) and appeal to the polynomial-time simulation of a RAM/assembly-program by a TM as covered earlier in the course. As a function of the input length $N$ (not $n$), what is the asymptotic running time of your pseudocode? (This is how running time is measured in CSE531. 15+3 = 18 pts.)

(b) Now define $L_3$ to be the language of undirected graphs $G$ whose vertex set $V$ can be broken into $V_1 \cup V_2 \cup V_3$ such that all edges from $V_i$ go to $V_j$ where $j \neq i$. For example, the 5-vertex ”bowtie graph” $G$ with $E(G) = \{(1,2),(2,3),(1,3),(3,4),(3,5),(4,5)\}$ belongs to $L_3$, but the complete graph on 4 vertices does not. Show that $L_3$ belongs to NP. (Think about why your approach in the case of $L_2$ fails, but don’t spend any time trying to patch it into a polynomial-time algorithm here. 12 pts.)

(3) A short-essay answer

We refer to the proof of the Cook-Levin Theorem that was given in class. Suppose we were to negate every variable in every block of three clauses for incoming wires $u,v$ and outgoing wire $w$ from a gate $g$, so that it now reads:

$$(\bar{u} \lor \bar{v}) \land (\bar{v} \lor \bar{w}) \land (u \lor v \lor w).$$

But suppose we still require that the output wire $w_0$ outputs 1 for acceptance. Is the construction still correct—or correctable? And please show exactly what would happen if we were to use the other universal gate, NOR, in place of NAND in the construction of $\phi_n$ in the proof. (18 pts.)

(4) “Choosing an Administration.”
The President-elect needs to fill a fairly large number of federal positions—many more besides the Cabinet—from an even larger number $n$ of qualified people. He/she also listens to various advisors and special-interest groups, who submit a large number $m$ of positive and negative recommendations. A positive recommendation lists 1 or 2 or 3 or more people and says, “choose at least one of these guys.” A negative recommendation gives a similar list but says, “Don’t choose all of these guys—that would be favoring this group too much.” The recommendations don’t necessarily specify a particular federal position—they only talk about whether particular people $x$, $y$, and/or $z$ should belong to the Administration in general.

Consider the task of choosing whether each person should belong to the Administration. Can this be done in a way that meets every recommendation? Prove that the decision problem framed by this question is NP-complete, preferably by reduction from (3)SAT and in any event by showing how this problem is “SAT-like.” (Thus in a formal, computational sense it is hard to form a government. 21 pts.)

(5) Show that the following decision problem is NP-complete:

**Dominating Set**

**Instance:** An undirected graph $G = (V, E)$ and an integer $k$, $1 \leq k \leq |V|$.  

**Question:** Does there exist a subset $U \subseteq V$ of size at most $k$ such that every other vertex in $V$ is adjacent to one in $U$ (that is, $(\forall v \in V \setminus U)(\exists u \in U) : (u, v) \in E$)?

Needless to say, it is forbidden to look this up on the Internet.

For some helpful examples, if $G$ is a “triangle,” any one node forms a dominating set that is not a vertex cover. If $G$ has 6 vertices and edges in the form of the letter ‘H’ then the only dominating set of size 2 is obtained by choosing the two middle vertices, and these do not form an independent set. (24 pts., for 111 on the set)