Def **A**. An NFA $N$ can process a string $x \in \Sigma^*$ from state $p$ to state $q$ if there is a sequence $(q_0, U_1, \ldots, q_{m-1}, U_m, q_m)$ such that:

- $q_0 = p$ and $q_m = q$,
- for all $i, 1 \leq i \leq m$, $(q_{i-1}, U_i, q_i)$ is in $\delta$.
- The sequence is called a (valid) computation (path).

Def: $L_p = \{x \in \Sigma^* : N$ can process $x$ from $p$ to $q\}$, and finally formally:

$\mathcal{L}(N) = \bigcup_{q \in F} L_q = \{x \in \Sigma^* : N$ can process $x$ from some start state $q \in F\}$.

Example: $N = \epsilon$.

Is $x = 1011$ in $L(N)$?

No: $(5, 1, 8, 0, 8, 1, 8) \ldots$ cannot be validly completed.

State $f$.

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Def **B**. The concatenation of two languages $A, B \subseteq \Sigma^*$ is defined by:

$A \cdot B = \{xy : x \in A \land y \in B\}$.

Translate English: “and then $y$”.

Does $A \cdot A = \{x \cdot x : x \in A\}^*$?

No: Common error.

$A \cdot B = \{z : z$ can be broken as $z = x \cdot y$ s.t. $x \in A \land y \in B\}$.

$A \cdot A = \{z : z$ can be broken as $z = x \cdot y$ s.t. $x \in A \land y \in A\}$.

$A = \{0, 01\}$

$A \cdot A = \{00, 01, 010, 0101\}$

$B = \{0, 10\}$. Also note:

$A \cdot B = \{00, 010, 0101, 0110\}$.

Always $A \cdot B \subseteq A \cdot |B|$, but $\|A \cdot B\| < |A| \cdot |B|$ is possible.
Picking up in the inductive cases of the proof, given regulars \( \alpha, \beta \) and equivalent NFAs \( N_\alpha, N_\beta \).

- \( Y = L_\ast (\beta) \) is a regex,
  
  \[ L(Y) = L(\beta) \cdot L(\beta) \cdot L(\beta) \cdot \ldots \]
  
  and given \( N_\alpha,N_\beta \), build \( N_Y \) =

  we need to prove \( L(N_Y) = L(Y) \). Not immediately obvious, but the flow:

  \( \begin{align*}
  A^* &= A^* \cdot A \cdot A^* \cdot A \\
  A \cdot B &= I \\
  A \cdot A &= I \\
  A \subseteq \emptyset \\
  B &= \emptyset \\
  \end{align*} \)

  By the particular series form of \( N_Y \), \( L(N_Y) = \{ w \in \Sigma^* : w \text{ can be broken as } w = x_1 y_1 \ldots x_n y_n \text{ and } L(N_{y_1}), \ldots, L(N_{y_n}) \} \)

  Induction hypothesis:

  about correctness of \( \Delta \) = \( \{ w \in \Sigma^* : w \text{ can be broken as } w = x_1 y_1 \ldots x_n y_n \text{ and } L(N_{y_1}), \ldots, L(N_{y_n}) \} \)

  the NFAs \( (N_\alpha \cdot L(\beta)) \cdot L(\beta) \) = \( A(Y) \cdot L(\beta) \), as needed to propagate the induction hypothesis.

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Equivalent NFAs \( N_\alpha, N_\beta \).

\( \gamma = \gamma \), \( L(\gamma) = L(\alpha)^* \) defined via

\[ \begin{align*}
A^* &= A^* \cdot A_1 \cdot A_2 \cdot A_3 \cdot \ldots \\
A \cdot B &= I \\
A \cdot A &= I \\
\emptyset \subseteq \emptyset \\
\emptyset &= \emptyset \\
\end{align*} \]

Feedback Circuit (with regex) \( \Box \)

\( \begin{align*}
A \cdot B &= \{ w : w \text{ can be broken as } w = x_1 y_1 \ldots x_n y_n \text{ and } L(N_{y_1}), \ldots, L(N_{y_n}) \} \\
A \cdot A &= \{ w : w \text{ can be broken as } w = x_1 y_1 \ldots x_n y_n \text{ and } L(N_{y_1}), \ldots, L(N_{y_n}) \} \\
\emptyset \subseteq \emptyset \\
\emptyset &= \emptyset \\
\end{align*} \)

\( \begin{align*}
A \cdot B &= \{ w_0, 0, 0, 0, 1, 0, 0, 1, 0, 1 \} \\
A \cdot A &= \{ w_0, 0, 1, 0, 0, 1, 0, 0, 1, 0 \} \\
\emptyset \subseteq \emptyset \\
\emptyset &= \emptyset \\
\end{align*} \)

\( \begin{align*}
A \cdot B &= \{ w_0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0 \} \\
A \cdot A &= \{ w_0, 0, 0, 0, 0, 1, 0, 1, 0, 1, 0, 0 \} \\
\emptyset \subseteq \emptyset \\
\emptyset &= \emptyset \\
\end{align*} \)

Always:

\( \| A \cdot B \| = \| A \cdot B \| \), but \( \| A \cdot B \| < \| A \cdot B \| \) is possible.