Def: The concatenation of two languages $A, B \subseteq \Sigma^*$ is $A \cdot B = \{xy : x \in A \land y \in B\}$

Example: $A = \{0, 01\}$, $B = \{0, 10\}$

$A \cdot B = \{00, 010, 0110\}$

Compare: $A = \{00, 010, 0110\}$

$A \times B = \{(0, 0), (0, 10), (01, 0), (01, 10)\}$

Note: Different as pairs is possible.

Always $|A \times B| \leq |A| \cdot |B|$, but $|A \cdot B| < |A| \cdot |B|$

Is $A \cdot A = \{x : x \in A\}$? No: $A \cdot A$ above $= \{00, 001, 010, 011, 100\}$

Rewrite the def as: $A \cdot B = \{\exists z \in \Sigma^* : z \text{ can be broken as } z = x \cdot y \text{ such that } x \in A \land y \in B\}$

Def: $A \cdot A = \exists z$; $z$ can be broken as $z = x \cdot y$ where $x \in A$ and $y \in A$.

$L^0 = 1$

$A^0 = \emptyset$. By convention, $A^n = \emptyset$ for all $A$, even $\emptyset$.

Def: $A^\ast = \cup_{m=0} A^m = \{\emptyset, A, A^2, A^3, \ldots\}$. Kleene Star.
Rejoining the proof of (the first leg of) Kleene's Theorem

Induction case 5. Given regexps \( \alpha \) and \( \beta \). By Ind. Hyp we may take NFAs, \( N_\alpha, N_\beta \) such that 
\[ L(N_\alpha) = L(\alpha), \quad L(N_\beta) = L(\beta), \]
and we build \( N_\gamma \) from \( N_\alpha, N_\beta \) like so:

\[
\begin{align*}
N_\gamma : \\
\quad s_\gamma \xrightarrow{\varepsilon} s_\alpha \xrightarrow{\varepsilon} s_\beta \xrightarrow{\varepsilon} s_\gamma
\end{align*}
\]

Need to prove: 
\[ L(N_\gamma) = L(N_\alpha) \cdot L(N_\beta) \] so 
\[ L(\gamma) = L(\alpha) \cdot L(\beta) \] 

For any string \( z \in Z^* \), this is our first step. 

\( z \in L(N_\gamma) \) if \( z \) can process \( z \) from \( s_\gamma \) to \( s_\gamma \). By diagram

\( z \) can be broken as \( z = x \cdot y \) such that \( N_\alpha \) can process \( x \) from \( s_\alpha \) to \( s_\beta \) and \( N_\beta \) can process \( y \) from \( s_\beta \) to \( s_\gamma \).

\( z \) can be broken as \( z = x \cdot y \) such that \( x \in L(N_\alpha) \) and \( y \in L(N_\beta) \).

\( z \in L(N_\alpha) \cdot L(N_\beta) \), so 
\[ L(N_\gamma) = L(N_\alpha) \cdot L(N_\beta) \] 

By I.H., 
\( L(N_\gamma) = L(\alpha) \cdot L(\beta) \), which = \( L(\gamma) \)

\( i \) \( L(N_\gamma) = L(\gamma) \) is proved by defn of \( L(\gamma) \).
6. Given just $\lambda$ and $N_{\lambda}$ st. $L(N_{\lambda}) = L(\lambda)$, we define $\gamma = \alpha^*$ to have the semantics

$$L(\gamma) = [L(\alpha)]^*.$$  

Build $N_{\gamma}$ as:

Then $N_{\gamma}$ can process a string $z$ from $S_{\gamma}$ to $F_{\gamma}$ iff $z$ can be broken as $z = z_1 \cdot z_2 \cdots \cdot z_m$ such that for each $i$, $1 \leq i \leq m$,  

for each $i$, $z_i \in A^*$, for each $i$, $z_i \in A^*$, we get $L(N_{\gamma}) = L(N_{\lambda})$  

by IH $= L(\alpha)^* = L(\gamma)$.  

This also completes both the formal inductive definition of regular expressions $\gamma$ and the proof that some NFA $N_{\gamma}$ gives $L(N_{\gamma}) = L(\gamma)$.

Without the blue $\epsilon$-arc we get: $L(N_{\lambda})^+ = L(N_{\lambda}) \cup L(N_{\lambda})^2 \cdots$ instead.