Let \( L \) be a language we "hope" is regular. Let \( x \) and \( y \) be two strings in \( \Sigma^* \) \( (x+y) \)
Suppose there is a string \( z \in \Sigma^* \) such that
\( xz \in L \) but \( yz \notin L \) or vice-versa,
\( \text{i.e. } xz \in L \) and \( yz \in L \).

\[ L(xz) \neq L(yz) \]

Then any possible DFA \( M = (Q, \Sigma, \delta, s, F) \) s.t. \( L(M) = L \)
must process \( x \) and \( y \) to different states \( p, q \)
from \( s \)

Write \( x \equiv_L y \) if for all \( z \in \Sigma^* \), \( L(xz) = L(yz) \)

Then \( x \equiv_L y \) means \( \forall z \in \Sigma^* \) \( L(xz) = L(yz) \)
as above.

Note: \( x \equiv_L y \)
For any \( L \), \( x \equiv_L y \leftrightarrow y \equiv_L x \) because the condition is symmetric.
and \( w \equiv_L x \land x \equiv_L y \Rightarrow \forall z \in \Sigma^* \)
\( L(wz) = L(xz) \land L(xz) = L(yz) \)
\( \Rightarrow \forall z \in \Sigma^* \), \( L(wz) = L(yz) \), \( i.e. \Rightarrow w \equiv_L y \).
Thus $\equiv_L$ is an equivalence relation on $\Sigma^*$.

It partitions $\Sigma^*$ into equivalence classes. There might be finitely or $\omega$-many equivalence classes. $x \neq_L y$ means $x$ and $y$ are in different classes.

Mullin-Nerode Theorem (1958, independently by John Myhill and Anil Nerode, Cornell)

A language $L$ is regular $\iff$ the relation $\equiv_L$ has finitely many equivalence classes.

Proof: $\Rightarrow$ states: if $\equiv_L$ has $\omega$-many equivalence classes, then $L$ is not regular.

By $L$ having $\omega$-many equivalence classes, we can make an infinite set $S$ by choosing one string from each class $S$ has the property: $(\forall x, y \in S \forall u \exists y \neq_L u) x \neq_L y$.

Suppose there were a DFA $M$ s.t. $L(M) = L$. It would have some finite number $K$ of states. But $S$ has more than $(\operatorname{any}) K$ strings, and so $M$ would be forced: $M$ must send some pair $x, y \in S$ to the same state. Contradiction.
Proof, part 2 ($\Leftarrow$): if $L$ has finitely many equiv. classes $\rightarrow L$ is regular

For every equivalence class $[x]_L$, we can think of a least string in the class as its shortest name.

Define $M = (Q, \Sigma, \Delta, s, F)$ where

$Q = \{\text{equiv. classes}\}$, $s = [\varepsilon]$, $F = \{[x] : x \in L\}$

and for all $c \in \Sigma$ and $[x] \in Q$ where $x$ is "the name"

$\Delta([x], c) = [xc]$

Then $L(M) = L$.

"The name" might have a lesser name but it's the same class.

And if there are only finitely many equiv. classes, then $M$ really is a DFA s.t. $L(M) = L$,

so $L$ is regular. □

Self-study: convince yourself that for all $x, y, c$

$[xc] = [yc] \iff [x] = [y]$

This says that "$\Delta$ is well-defined?"