Defn: A language $A$ is recursive if there is a DTM $M$ such that $L(M) = A$.

A language $A$ is computably enumerable (c.e) if there is a DTM $M$ such that $L(M) = A$.

If also $M$ is total, i.e., halts for all inputs $(\forall x \ M(x) \downarrow)$, then $A$ is Turing-acceptable.

A TM $M$ runs in time $t(n)$ if for all $x \in \Sigma^*$, the computation $M(x)$ halts within $t(\|x\|)$ steps.

A TM $M$ runs in space $s(n)$ if $\forall x$, all computations on $M$ use space at most $s(\|x\|)$ steps.

$\text{DTIME}[t(n)] = \{ A : A = L(M) \text{ for some DTM } M \text{ that runs in time } t(n) \}$

$\text{NTIME}[t(n)] = \{ A : A = L(N) \text{ for some NTM } N \}$

$\text{DSPACE}[s(n)] = \{ A : A = L(M) \text{ where } M \text{ is a DTM that runs in space } s(n) \}$

$\text{NSPACE}[s(n)] = \{ A : A = L(N) \text{ for some NTM } N \}$
Generally we postulate that a time function $t(n)$ satisfies $t(n) \geq n^k$, and a space function $S(n)$ is either constant or satisfies $S(n) \geq \frac{1}{e} \log_2 n$ for some $C > 0$. Let's say...

**Important Cases:**

- **$DLIN = OTIME[O(n)]$**
- **$NLIN = NTIME[O(n)]$**
- **$P = \cup_{k \geq 1} OTIME[O(n^k)]$**
- **$NP = \cup_{k \geq 1} NTIME[O(n^k)]$**

$PSPACE = \cup_{k \geq 1} OSPACE[O(n^k)]$

$NPSPACE$

$L = OSPACE[O(\log n)]$

$NL = NSPACE[O(\log n)]$

**$EXP = \cup_{k \geq 1} OTIME[2^{O(n^k)}]$**

$DSPACE[O(n)] \subseteq \text{Reg} \subseteq \text{ATIME}(n \log n) \subseteq DLIN$

Deterministic linear time on a multi-tape TM can be single-tape (1) ignoring why $= \{L(M) : M \text{ is a TM that runs in some time } T(n) \text{ that is } \leq a \cdot n^k + b \text{ for some } a, k \geq 1, \forall n \geq n_0\}$

"Non-deterministic Polynomial Time"

"Polynomial Space"

Next month: $= PSPACE$! deterministic log space $L = NL$ unknown

"At the pole" Exponential Time

There is also $NEXP$ and more.
"Cone Diagram" of all these classes:

Convention:
Classes are
include downward:

REC $\subseteq$ RE,
NP $\subseteq$ PSPACE
$P \subseteq NL \subseteq L \subseteq RE$

$P \neq$ EXP
PSPACE is also unknown
$L \neq$ NP is unknown
BQP is there

Not only is $P \neq$ NP unknown
$L \neq$ NP is unknown

Starting at the top end, the diagram communicates these relationships:

REC $\not\subseteq$ RE, co-RE$\not\subseteq$ RE, but RE$\subseteq$ RE $\subseteq$ RE.

Note for contrast: $P \subseteq$ NP, $P \subseteq$ coNP, but NP $\cap$ coNP is not known to $\equiv P$.

REC is closed under complements.

co-REC = REC, ie L decidable $\Rightarrow$ $\bar{L}$ decidable

Proof of this fact: Let any $A \in$ REC be given.
Then there is a TM $M_A = (Q_A, \Sigma, \Gamma, \delta, q_{acc}, q_{rej})$ s.t. $L(M_A) = A$ and
for all $x$, $M(x) \downarrow$ and halts either in $q_{acc}$ or $q_{rej}, \Rightarrow A \in$ REC. $\Box$

Define $M' = (Q_A, \Sigma, \Gamma, \delta', q_{acc}', q_{rej}')$. Then $L(M') = \overline{A}$ and $M'$ is also total.