**Defn:** A and B are many-one equivalent, written $A \equiv_m^p B$ if $A \leq_m^p B$ and $B \leq_m^p A$. Note: $\leq_m$ and $\leq_m^p$ are transitive, so $A \equiv_m B$ and $A \equiv_m^p B$ are equivalence relations.

Example: $K_m \leq_m A_{TM}$ by restriction and $A_{TM} \leq_m K_m$ by all-one switch.

**Theorem:** For every language $A \in \text{RE}$, $A \leq_m A_{TM}$.

**Proof:** Given $A$, we can take a TM $M_a$ such that $L(M_a) = A$. Then for all $x \in \sum^*$,

- Define $f(x) = \langle M_a, x \rangle$.
- Then $x \in A \iff M_a \text{ accepts } x \iff \langle M_a, x \rangle \in A_{TM}$.

Thus $K_m \equiv_m f \circ A_{TM}$, so the reduction is Correct.

**Corollary:** If $A$ is complete for $\text{C-RE}$, then $A \leq_m A_{TM}$.

**Defn:** A language $B$ is hard for $\text{RE}$ under $\leq_m$ if for all $A \in \text{RE}$, $A \leq_m B$. It is also $\text{RE}$-complete if $\text{RE} \subseteq \leq_m B$.

If for all $A \in \text{RE}$, $A \leq_m B$, then $B$ is $\text{RE}$-complete.

Types of languages $B$: $B \equiv_m A_{TM}$ are $\text{RE}$-complete.

- Given any class $C$ and language $B$. It is hard for $C$ under $\leq_m$ if for all $A \in C$, $A \leq_m B$.

**Theorem:** If $A \equiv_m B$ then the class $\leq_m$ is the same as $\leq_m^p$.

If $A \equiv_m^p B$, then $A$ and $B$ are equivalent under $\leq_m^p$.

**Corollary:** If $A$ is $\text{RE}$-complete, then $A_{TM} \leq_m^p A$.

When $A_{TM}$ has a peak, $\leq_m$ is the same as $\leq_m^p$.

**Conclusion:** If $A$ is $\text{RE}$-complete, then $A_{TM}$ is $\text{RE}$-complete. Hence, $\text{RE}$ and $\text{RE}$ are equivalent under $\leq_m^p$.