For any languages $A, B \subseteq \Sigma^*$, write $A \leq_m B$ (m for "many-one" or "mapping") if there is a computable function $f: \Sigma^* \rightarrow \Sigma^*$ such that for all $y$, $x \in A \iff f(x) \in B$. If $f$ is polynomial-time computable we write $A \leq_p B$. If log-space computable, write $A \leq_{\log} B$ etc.

Note about the computation of functions.

- Input tape: read-only
  - $\Delta X_1 X_2 X_3 X_4 \ldots X_n$
- Work tapes: 
  - $\vdots$
  - $\vdots$
- Output "tape": one-way, write-only = a stream, not counted against space bound

Text identifies "streaming algo" with $O(\log n)$ space functions. Std. def allows $O((\log n)^m)$ for some $m$: "poly-log space."
Example: \( K_{\text{TM}} = \{ X : X \text{ codes TM } M \text{ s.t. } M \text{ accepts } X \} \)

Then \( A_{\text{TM}} = \{ \langle M, x \rangle : M \text{ accepts } x \} \).

\( K_{\text{TM}} \leq_m A_{\text{TM}} \text{ via the mapping } f(x) = \langle x, x \rangle \).

**NE_{\text{TM}}**: Instance: A DTM \( M \).

**Question**: Is \( L(M) \neq \emptyset \)?

As a language:

\( \text{NE}_{\text{TM}} = \{ \langle M \rangle : L(M) \neq \emptyset \} \).

Undecidable.

**Theorem**: \( \text{NE}_{\text{TM}} \) is c.e. but \( A_{\text{TM}} \leq_m \text{NE}_{\text{TM}} \) so it is c.e.

**Proof**: "Is c.e. := Imagine a NFA \( N \) that on input \( \langle M \rangle \) guesses a string \( x \) and accepts it if and when \( M \) accepts \( x \). Runs \( M(x) \), then \( L(N) = \text{NE}_{\text{TM}} \) can convert \( N \) to DTM, so \( \text{NE}_{\text{TM}} \) is c.e.

**Reduction**: Map from \( A_{\text{TM}} \), instances \( \langle M, x \rangle \)

Map to a single machines \( \langle M' \rangle \) which are instances of the \( \text{NE}_{\text{TM}} \) problem.

Then \( \langle M, x \rangle \in A_{\text{TM}} \iff M \text{ accepts } x \iff \langle M' \rangle \in \text{NE}_{\text{TM}} \iff M' \text{ accepts every } y \iff L(M') = \Sigma^* \implies L(M') \neq \emptyset \). But \( \langle M, x \rangle \in A_{\text{TM}} \iff \forall y, M'(y) \text{ never can accept } \iff f(X, x) \in \text{NE}_{\text{TM}} \iff A_{\text{TM}} \leq_m \text{NE}_{\text{TM}} \). So \( A_{\text{TM}} \leq_m \text{NE}_{\text{TM}} \).

**Simulate \( M(x) \)** if and when \( M \) accepts \( x \)

**Accept \( y \).**
Convention: The relation $A \leq_m B$ is diagrammed by having the angle from $A$ up to $B$ be at least as steep as the cone walls.

Monday Preview: This will follow from the "Reduction Theorem": Suppose $A \leq_m B$.

Then:

- If $B$ is decidable, then $A$ is decidable.
- $B$ is c.e. $\Rightarrow$ $A$ is c.e.
- $B$ is co-c.e. $\Rightarrow$ $A$ is co-c.e.

And when we know $A \leq_P B$:

- $B \in P \Rightarrow A \in P$
- $B \in \text{coNP} \Rightarrow A \in \text{coNP}$
- $B \in \text{NP} \Rightarrow A \in \text{NP}$
- $A \in \text{NP}$