Reduction Meta-Theorem: Suppose $B$ is in P time decidable and $A \leq_p B$. Then $A$ also is decidable.

Proof: Take a $p(n)$-time machine $M_B$ such that $L(M_B) = B$.

Build a machine $M_A$ for $A$ as follows, using a total machine $T$ computing the reduction $f$. By def of $A \leq_m B$ via $f$,

Then for all $x \in \Sigma^*$:

$\forall x \in A \iff f(x) \in B$

We deduced this.

$M_A \text{ accepts } x \iff M_B \text{ accepts } y = f(x)$

Hypothesis: $L(M_B) = B$

$\therefore L(M_A) = A$.

Thus $A$ is decidable.

How can we bound the running time of $M_A$? Not only could $T(f(x))$ take $O(|x|^2)$ time, it could output $y$ of length $n'$:

$|y| = n' = n^2$.

$M_B$ runs in time $O(\ell y)^K = O(n'^K) = O(f(n^2)^K) = \text{time } O(n^{K\ell})$. Still a (bigger) polynomial time!
**Corollary:** If \( \overline{B} \) is c.e. and \( A \leq_m B \) then \( A \) is c.e.

**Proof:** \( B \vdash B \) being c.e., we can take a TM \( M_B \) s.t. \( L(M_B) = \overline{B} \). (\( B \) is c.e.)

Theorem gives a TM \( M_A \) s.t. \( L(M_A) = \overline{A} \). because

\[ A \leq_m B \implies \exists f \text{ s.t. for all } x \in \Sigma^*, \ x \in A \iff f(x) \in B. \]

Applications come from the contrapositive:

- Suppose \( A \leq_m B \).
  - If \( A \) is undecidable then \( B \) is undecidable.
  - If \( A \) is not c.e. then so is \( B \).
  - If \( A \) is not co-c.e., then \( B \) is not either.
"The" Halting Problem: INST: A TM $M$, and an input $y$ to $M$

$H_P^M = \{(M, y): M(y) \downarrow\}$  GUESS: Does $M(y) \downarrow$?

Show $AP_{TM} \leq_m H_P^M$, so $H_P^M$ is undecidable.

Reducible function $f$ needs to map

$\langle M, x \rangle$ to $\langle M', y \rangle$ st.

$M'(y)$ halts when and only when $M(x)$ accepts.

Do with $y = x$ (but we'll generically on Friday)

$\langle M, x \rangle \mapsto \langle M', x \rangle$ $M' =$

If $M(x)$ halts w/o accepting, $M'(x)$ doesn't halt at $y$.

If $M(x)$ doesn't halt,

$M'(x)$ doesn't halt either. So the traditional Halting Problem is undecidable, indeed its language is (c.e. but) not p-c.e.