Show $\text{ALLTM} \leq_m \text{TOT}$. We need to give a computable function $f : \{\text{codes of TMs}\} \rightarrow \{\text{codes of TMs}\}$ such that for all codes $<M>$, $f(<M>)$ is the code of a TM $M'$ such that $<M> \in \text{ALLTM} \iff <M'> \in \text{TOT}$.

I.e. $<M> \in \text{ALLTM}$ if and only if $\forall x (M'(x) \downarrow \iff L(M) = \Sigma^*)$.

Major Points: $f$ is a simple task of adding a loop $(c/s)$ at $q_{rel}$ for all $M$.

Construction: $f(M) = \text{Sim } M(x)$

- Halt and accept $(c/s)$ if $M$ halts and rejects
- Don't halt.

If we demand that $M'$ has a $q_{rel}$ state, it can be elsewhere $q_{rel}$ of $M$ and isolated.

Computable: $f$ is a simple task of adding a loop $(c/s)$ at $q_{rel}$ for all $M$.

In fact $f$ is streamable in linear time, so $\text{ALLTM} \leq^* \text{TOT}$ too.

Correctness:

- For all arguments $<M>$ to $f$:
  - $<M> \in \text{ALLTM} \iff L(M) = \Sigma^*$
  - $\forall x (M'(x) \downarrow \iff (\forall x) M(x) \text{ accepts}$
  - $\Rightarrow (\forall x) M'(x) \downarrow \iff <M'> \in \text{TOT}$.

- Either $M(x)$ goes to the $q_{rel}$ $f_M \Rightarrow M'(x) \uparrow$, indeed, loops in $\text{rel}$ for $M(x)$ doesn't halt $M'(x) \uparrow$ also because it otherwise simulates $M(x)$.

- $f$ is a correct mapping read.
Show $\text{TOT} \leq_m \text{ALL}_T$. We need a function $g : \{\text{TM codes}\} \to \{\text{TM codes}\}$ st. for all $M$, $M$ is total $\iff g(M) \in \text{ALL}_T$.

**Construction:** Map $\langle M \rangle \to g(M)$. Let $M'' = g(M)$.

Input $x$:

- Simulate $M(x)$ if and when it stops.
- Take of $M$.
- For correctness, we want $\forall x, M(M''(x)) \Downarrow = \forall x, M''(x)$ goes either to $\downarrow$ or to $\downarrow$ (accepts).
- Hence, $M''(x)$ is $\in \text{ALL}_T$.

**Complexity:** $g$ is again a simple code edit (linear time).

So we got $\text{ALL} \leq_m \text{TOT}$ and $\text{TOT} \leq_m \text{ALL}_T$. We write $\text{ALL} \equiv_m \text{TOT}$.

In fact, we got $\text{ALL} \equiv_p \text{TOT}$ & $\text{TOT} \equiv_p \text{ALL}_T$. So $\text{ALL} \equiv_m \text{TOT}$.

Similarly $\text{HP} \equiv_p \text{HP} \equiv_m \text{HP}$, so $\text{HP} \equiv_m \text{HP}$.

These are equivalence relations, since $\leq_m$ and $\leq_p$ are reflexive and transitive.

**Def:** A language $B$ is hard for a class $C$ of languages under a reducibility $\leq_r$ if for all $A \in C$, $A \leq_r B$.

If also $B \in C$, then $B$ is complete for $C$ (under the reducibility).

**Thm:** For all $A \in \text{RE}$, $A \leq_p \text{HP}$. (Note: $A \notin \text{RE}$.)

$\text{HP} \equiv_m \text{HP}$, so $\text{HP}$ is like $\text{RE}$ complete.