Last time: $A_{TM} \leq_{m} \overline{ALL_{TM}}$, so $ALL_{TM}$ is not co-c.e.
Show $D_{TM} \leq_{m} ALL_{TM}$, so $ALL_{TM}$ is not c.e.

Try 3: "Delay Switch".
Need $\langle M \rangle \rightarrow \langle M' \rangle$ st.

$\langle M \rangle \in D_{TM} \Rightarrow \langle M \rangle$ does not accept its own code $\Rightarrow \text{LIN' } = \Sigma$.

$\langle M \rangle \notin D_{TM} \Rightarrow (M) \text{ does accept } L(M) \Rightarrow L(M') \neq \Sigma$.

Construction:
Given any TM $M$,
Map $M \rightarrow M' = \langle M \rangle$.

Key "delay" point: If $M$ does accept $\langle M \rangle$ it does so in some number $t$ of steps, so that whenever $|X| = n \geq t$, this acceptance is discovered and $X$ is rejected by $M$.
So $\langle M \rangle \notin D_{TM} \Rightarrow (3t) \text{ L}(M) \subseteq \{0, 1\}^* \Rightarrow L(M')$ is finite $\Rightarrow L(M') \neq \Sigma$.

NAME: \text{Give s: Is } L(M) \text{ infinite?}

Yes: $\text{LIN} \leq_{m} \text{ INF}$, the "Index-Set" of the class of infinite c.e. languages.
**Definition:** For any subclass $C$ of the r.e. languages, its index set is $I_C = \{ \langle M \rangle : L(M) \in C \}$.

As a problem, it is: $\text{INST: A TM } M \quad \text{Ques: Is } L(M) \text{ in } C$.

**Examples:**
- $\text{ETM}$ is the index set $I_{\text{ETM}} = \{ \langle M \rangle : L(M) \neq \emptyset \} = I_{\{ \emptyset \}^c}$.
- $\text{NE} = \{ \langle M \rangle : L(M) \neq \emptyset \} = I_{\{ \emptyset \}^c} \cap \text{TM}$.
- $\text{ALL} = I_{\Sigma^*}$.
- $\text{IN} = \{ \langle M \rangle : L(M) \text{ is infinite} \}$.
- $\text{REC} = \{ \langle M \rangle : M \text{ is a TM and } L(M) \text{ is regular} \}$ which includes $\text{ETM}$.
- $\text{IP} = \{ \langle M \rangle : L(M) \in \mathbb{P} \}$ irrespective of whether $M$ is TMs that run in polynomial time.

What is $I_\emptyset$, the index set of the empty class?
What is $I_{\text{REC}}$, the index set of the class of all r.e. languages?

$I_{\text{REC}} = \mathbb{N} = \Sigma^*$. So $I_\emptyset$ is the complement of $\Sigma^*$, i.e., $I_\emptyset = \emptyset$.

**Rice's Theorem:** Those are the only two decidable index sets.

"Moral" (in the version extended to classes of functions): Every nontrivial extensional property of programs is undecidable.
Proof: Given \( L \) when \( C \neq \mathcal{O} \) and \( C \neq \mathcal{RE} \), so there is a c.e. language \( A \) such that either:

\( \text{a) } \emptyset \in \mathcal{C}, \ A \notin \mathcal{C} \) \hspace{1cm} (\ast) 

or \( \text{b) } A \in \mathcal{C}, \ \emptyset \notin \mathcal{C} \).

Take an \( M_A \) s.t. \( L(M_A) = A \).

Build \( (M, w) C \rightarrow M' \rightarrow X \)

\[ \text{Sim} \ M(w) \]

if and when it accepts

\[ \text{Run} \ M_A(X) \] if \( M_A \) does.

In case (\( \ast \)):

\( (M, w) \in \mathcal{A}_M \rightarrow L(M') = A \Rightarrow M' \notin \mathcal{C} \) \hspace{1cm} \text{\( A_m \notin \mathcal{C} \)}

\( (M, w) \notin \mathcal{A}_M \rightarrow L(M') = \emptyset \Rightarrow M' \in \mathcal{C} \)

Either way, \( \mathcal{C} \) is undecidable. \( \Box \)

Example: \( C = \mathcal{REG} \), case (\( \ast \)) applies with \( A = \{ \text{palindromes} \} \), so \( \mathcal{REG} \) is undecidable.

Added:

I could have worded this as, “Given any class \( C \), first suppose (\( \ast \)) that the empty language \( \emptyset \) is in \( C \). E.g., \( C \) is the class of regular languages. Then by \( C \neq \mathcal{RE} \), there is some c.e. language \( A \) that is not in \( C \). Take an \( M_A \) s.t. \( L(M_A) = \emptyset \). e.g., \( A = \{ \text{palindromes} \} \). If (\( \ast \)) \( \emptyset \) is not in \( C \), then by \( C \neq \mathcal{RE} \) we can take some \( A \) that is in \( C \), take \( M_A \) s.t. \( L(M_A) = A \), and continue as above.