A_DFA = \{ \langle M, x \rangle : M \text{ is a DFA and } M \text{ accepts } x \}.

Decidable: In \( p \): just run \( M(x) \).

Time: \( O(N \cdot m) \) if the code of \( M \) has to be accessed linearly.

Even on a TM, both \( M, N \leq N = |\langle M \rangle| \), so \( m \cdot n = O(N^3) \); quadratic time.

A_NFA = \{ \langle N, x \rangle : N \text{ is an NFA and } N \text{ accepts } x \}.

Decidable:

- DeBray (58) says: Convert \( N \) to DFA \( M \) and run as for \( \langle M, x \rangle \).

Flaw: this approach can use \( \exp(N) \) time. Time \( \propto N \cdot \exp(N) \).

Maintain for \( i = 0, \ldots, N = |x| \) the sets \( S_i \) of states that \( N \) can possibly process \( x_i \) to. Then \( x = L(N) \Rightarrow S_n \cap F \neq \emptyset \).

Time:

\( |N| \cdot \text{(update up to } m \text{ entries of } S_{i-1} \text{ to } S_i) \sim O(mn) \) in a given random access. Direct TM time: \( O(N^3) \).

\( E_{DFA} = \{ M : M \text{ is a DFA and } L(M) = \emptyset \} \)

\( N_{DFA} = \{ M : M \text{ is a DFA and } L(M) \neq \emptyset \} \)

\( N_{NFA} = \{ N : N \text{ is an NFA and } L(N) \neq \emptyset \} \).

In \( \mathbb{P} \) by breadth-first search \( \text{(BFS)} \) in the graph of \( M \) or \( N \).

\( E\text{DFA} = \sim N_{DFA} \) so \( \text{it is in } \mathbb{P} \) too.

there is a path from \( s \) to some state \( f \in F \).

Whatever string \( x \) is processed by the path belongs to the language.
4. \[ \text{ALL}_{\text{DFA}} = \{ \langle M \rangle : \text{M is a DFA and } L(M) \neq \Sigma^* \} \notin \text{E}_{\text{DFA}} \]

Algorithm: Complement \( M \rightarrow M' \) s.t. \( L(M') = \overline{L(M)} \).
Then \( L(M) = \Sigma^* \Leftrightarrow L(M') = \emptyset \Leftrightarrow \langle M' \rangle \in \text{E}_{\text{DFA}} \Leftrightarrow \langle M' \rangle \notin \text{NE}_{\text{DFA}} \).
This gives a \( \leq_P \) reduction from \( \text{ALL}_{\text{DFA}} \) to \( \text{E}_{\text{DFA}} \in \text{P} \), so \( \text{ALL}_{\text{DFA}} \notin \text{P} \).

5. \[ \text{ALL}_{\text{NFA}} = \{ \langle N \rangle : \text{N is an NFA and } L(N) \neq \Sigma^* \} . \]

\underline{Shack:} Not even known to belong to NP or to co-NP!
Complement \( \overline{\text{ALL}_{\text{NFA}}} = \{ \langle N \rangle : L(N) \neq \Sigma^* \} \) is NP-hard.

6. \[ \text{ALL}_{\text{SHORT}}_{\text{NFA}} = \{ \langle N, K \rangle : K \leq m = \| \langle Q, N \rangle \| \text{ and } \emptyset, 1, 2^K \subseteq L(N) \} . \]
   - N accepts all strings of "short" length K.
   - If the answer is NO, then we can guess a bad \( x \in \Sigma^* \leq K \) and verify that \( \langle N, x \rangle \notin \text{ANFA} \) using our poly-time routine
   (But if yes, we might have to try \( 2^K \) different x-es.)

The second shock with \( \text{ALL}_{\text{NFA}} \) is that the shortest \( x \in L(N) \) can have length exponential in the number of states of N! Too long for \( x \) to be an "NP witness" for the complementary language.

\[ \text{ALL}_{\text{SHORT}}_{\text{NFA}} \in \text{NP} \]

\[ \text{ALL}_{\text{SHORT}}_{\text{NFA}} \in \text{co-NP} \]
\textbf{SAT} = \{ \text{Boolean formulas } \phi \text{ using } \land, \lor, \neg \text{ and variables } x_1, \ldots, x_n, \text{ such that some truth assignment } a_1, \ldots, a_n \in \{0,1\}^n \text{ makes } \phi(a) = \text{true} \}.

\[ \{ \phi \} : (\exists a : |a| = n \leq |\phi| ) \left[ \phi(a) = \text{true} \right] \].

\begin{align*}
(\exists \text{linear } a) & \quad \text{in } O(n^2) \text{ time, actually } O(n) \text{ with random access.} \\
\Rightarrow \quad \text{SAT } \in \text{NP.}
\end{align*}

\text{TAUT} = \{ \phi(x_1, \ldots, x_n) : \phi \text{ is a tautology, i.e. } \left( \forall a \in \{0,1\}^n \right) \phi(a) = \text{true} \}.

\text{TAUT} = \{ \langle \phi \rangle : \neg \phi \notin \text{SAT} \} \not\subset \text{SAT}, \text{ in } \mathcal{O}(\text{-NP}).

We will in particular be concerned with formulas \( \phi \) that are conjunctions,

\[ \phi = C_1 \land C_2 \land \cdots \land C_m \]

of clauses, each of which is a disjunction of up to \( k \) literals.

A literal means a variable \( x_i \) or its negation \( \neg x_i \). A typical clause:

\[ C = (x_1 \lor \neg x_3 \lor x_6), \]

with \( k = 3 \) literals. This is called \( k \)-conjunctive normal form (\( k \text{-CNF} \)).

\textbf{Footnote: Debray's notes for Theorem 13.11 on page 43 give } O(n^2) \text{ time for } 3\text{SAT but for the wrong reason. The } 3\text{CNF} \text{ form is easily checkable in one pass in } O(n) \text{ time. The evaluation is linear only because one needs to make sure the assignment consistent and gives the same value to the same variable. But the formula can always be laid out to make it easy, so we can say both } \text{SAT and } 3\text{SAT belong to } \mathcal{N}L(\text{-LIN}) \text{ at } \mathcal{N}T(\text{-LIN}).