NP-completeness via $\leq^p_m$ reduction from 3SAT. Example:

INDEPENDENT SET

INST: An undirected graph $G$, and a number $K \leq |V| = n$.

QUEST: Is there a set $S \subseteq V$, $|S| \geq K$ such that no two nodes in $S$ are adjacent? (Extra: If yes, output optimal $S$ and $K$.

Example: Graph $G = \{1, 2, 3, 4, 5\}$

$K = 3$, answer is no: $S = \{3, 1, 2\}$, $|S| = 3$.

$K = 2$, answer is yes: $S = \{1, 2\}$ or $S = \{1, 5\}$.

$K = 1$, yes but not optimal (don't use node 3).

Theorem: INDEPENDENT SET is NP-complete: In NP and 3SAT $\leq^p_m$ INDEPENDENT SET.

Proof: "In NP": Given $G$, if $(G, K) \in$ INDEPENDENT SET then we can guess $S \subseteq V$.

$S \subseteq V$ means $|S| \leq |V| = n$ and length wise, $|S| \approx n \log n$.

Verif.: Check $(\forall u, v \in S, u \neq v) (u, v) \notin E$. $\therefore$ poly in $n \leq |G|$.

How many pairs to check? $(\binom{|S|}{2}) \leq (\frac{n}{2})^2 = O(n^2)$, polynomial in $n$.

INDEPENDENT SET $\in NP$.

Let $S \subseteq V$ with $|S| = K$. Then $\exists \phi \in \text{3SAT}$ with $\phi(S, K)$.

This is $\exists \phi \in \text{3SAT}$. $\therefore$ INDEPENDENT SET $\leq^p_m$ 3SAT.

Why can't we try all $S$-es? There are $2^{|V|}$ sub-sets $S \subseteq V$.

Hence trying them all leads to exponential time: $K = \frac{n}{2}$ very typical.

3SAT $\leq^p_m$ INDEPENDENT SET. Let any $\phi = C_1 \land C_2 \land \ldots \land C_m$ be given.

$\phi \iff \phi_1 \lor \phi_2 \lor \ldots \lor \phi_m$ where

$\phi_i = \phi(x_1, x_2, x_3) = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2 \lor x_3)$. \[\text{Let } m = \# \text{ clauses, each with (up to) 3 variables. } n = \# V.\]
Construction: Show how to build $G_{\Phi}$ and define $K_{\Phi}$ from $\Phi$.

Complexity: Say why the building is easy (often streamable).

Correctness: Show that if satisfiable $\iff G$ has an ind-set $S$ of size $K_{\Phi}$.

If not satisfiable $\iff$ all ind sets in $G$ have size $< K_{\Phi}$.

Idea: Set up a correspondence (not necessarily 1-1) between

sat. assgns $\langle a_1, \ldots, a_n \rangle \leftrightarrow \Phi$ and solutions $S$ to $G_{\Phi}$ for a critical value of $K_{\Phi}$.

"Variable Ladder and Clause Gadgets"

\[ X_1 \quad X_2 \quad X_3 \quad \ldots \quad X_n \]

Example: $n=3$

$K_{\Phi}$ = Idea: Put in "crossing edges" from clause gadget to rungs so that $S$ can have size $K_{\Phi}$ once from only if the choices of rung variables induce a sat. assgnt.

and many

Eg. choices induce the assignment (0,0,0) which satisfies $\Phi$ via $\overline{X_3}$ in $C_1$, $\overline{X_2}$ in $C_2$.

choices include $X_1 = 1$, whereas $X_2 = 0$, $X_3 = 1$ does not matter: $\Phi$ satisfied via $X_1$ in both clause

Blue lines are the "crossing edges" which depend on details of the particular $\Phi$.

The green "ladder edges" and "gadget edges" only depend on $n$ and $m$ from $\Phi$.

They define $K_{\Phi} = n + m$ regardless of details of $\Phi$ as the goal value. Next time: Formal details...