GRAPH 3-COLORING (3C) INST: A graph $G = (V,E)$ (undirected)

Ques: Can $G$ be 3-colored, i.e., is there a function

$\chi$ for

Chromatic $X: V \rightarrow \{R, G, B\}$ st:

$\forall (u,v) \in E, \chi(u) \neq \chi(v)$.

$\therefore 3C \in NP$.

$\forall \phi$ in $\Theta(m)$ time, $m \leq (2^n)^n$.

3SAT $\leq^P_\text{m} 3C$

First idea: establish a correspondence between possible colorings and truth assignments.

GREEN = TRUE 1
RED = FALSE 0
BLUE = Neutral, filler, not critical value

Example: $\phi = (x_1 \lor x_2 \lor \overline{x}_3) \land (x_1 \lor \overline{x}_2 \lor x_3)$

Correctness:

Any satisfying assignment allows at least one outer node in each clause gadget to be colored Red, which allows $G_{\phi}$ to be 3-colored. Vice-versa if $G_{\phi}$ has a 3-coloring then some outer node in each clause gadget must be red, and making the compatible running node Green thus gives an assignment that satisfies all clauses, so $G_{\phi} \in 3C \iff \phi \in \text{SAT}$. 

Subject to this interpretation, the crossing edges from $X_i$ in $G_i$ (node $x_i$), go to running node with the same sign, likewise $\overline{x}_i$ to $\overline{X}_i$. 

Fall 2018
**Q(13) Connecting Paths**

**INSTANCE:** A graph $G$ with some number $K$ of nodes $s_1, ..., s_K$ labeled "sources" and $K$ other nodes $t_1, ..., t_K$ as their "targets".

**QUEST:** Can we draw $K$ paths from each $s_i$ to its $t_i$ so that no two paths meet at any vertex?

**Case $K=1$:** Solve by BFS since no conflict is possible.

**Case $K=2$:** Impossible for $G =$ because the only possible paths conflict at the center node $c$.

Is it in $P$ for $K=2$? Hmmmm... $K=3$?

What about for any fixed $K$? Hmmmm...

We will show it is $NP$-complete when $K$ is allowed to vary. The idea is to take $K = N + M$ and dedicate $N$ $s_i$-$t_i$ pairs to truth assignments and the other $M$ pairs, capitalized as $S_j$-$T_j$, for the clauses $C_j$ in $\Phi$.

Example for $\Phi = (X_1 \lor \neg X_2 \lor X_3) \land (\neg X_1 \lor \neg X_2 \lor \neg X_3) \land (X_1 \lor \neg X_2)$:

Note that the "short clause" has only two horizontal paths.

The graph $G_\Phi$ and its distinguished source and target nodes can be "streamed" quickly from the clauses of $\Phi$, so the reduction $f(\varphi) = \langle G_\Phi, ..., \rangle$ is poly-time computable.

The satisfying assignment $X_1 = 0, X_2 = 0, X_3 = 1$ corresponds to the disjoint paths shown. Note how the first and third horizontal paths are forced, but we could choose $\overline{X}_{22}$ in the middle instead.