CSE 596 Lecture Wed Nov 7 Fall 2018

Time and Space Hierarchy Theorems

Ferry tape
size depends on \( x \)

Not mapped to a machine \( M' \) that depends on \( \langle M, x \rangle \) but input to one machine \( M' \)
3 tapes, binary alphabet plus blank

TM \( M \) with
- any \( K \) of tapes
- any size alphabet \( \Gamma \)
- any input \( x \)
- Given \( M \) is total:
  own time \( t_1(n) \)
  own space \( s_1(n) \)

Theorem (let in notes)
\( s_\nu(\langle M, x \rangle) \) can operate \( \begin{array}{c} \text{in} \quad \text{time} \quad \mu \text{min} \quad \text{in max} \end{array} \)

\[ t_1(n) \leq C_m \cdot s_1(n) + C_m \]

\[ s_1(n) \leq C_m \cdot t_1(n) \log t_1(n) \]

and \( s_2(n) \) is "reasonable"

Space Hierarchy Theorem: If \( s_1(n) = o(s_2(n)) \), then \( \text{SPACE}[s_1(n)] \neq \text{SPACE}[s_2(n)] \)

Time Hierarchy Theorem: If \( t_1(n) \log t_1(n) = o(t_2(n)) \) then \( \text{TIME}[t_1(n)] \neq \text{TIME}[t_2(n)] \)

\( t(n) = o(u(n)) \) means \( \lim_{n \to \infty} \frac{t(n)}{u(n)} \to 0 \)

\( C \quad \text{e} t(n) \leq u(n) \)

Example: \( t(n) = n \log n \) \quad ie. \quad \frac{t(n)}{\log n} = \log_{\frac{n}{\log n}} \)

\[ u(n) = \frac{1}{10000} \quad u'(n) = 0.0001 \quad \text{can prove using L'Hopital's Rule} \]

Simple example: \( n = o(n \log^2 n) = o(n^3) = o(n^3) = o(n^3) = o(2^n) \)

\( \text{P LIN} \subset \text{TIME}[n \log^2 n] \subset \text{DTIME}[n^2] \subset \text{DTIME}[n^3] \subset \text{P EXP} \)
Proof uses a special Diagonalizing Machine $M_s$:

- If $M_s$ or $M_t$:
  - Compute $t_2(n)$
- $M_s$ always runs within $S_2(n)$ tape cells of its own.

Suppose $L(M_s) \in \text{DSPACE}(S_2(n))$.

Then there would exist a DTM $Q$ that accepts $L(M_s)$ in space $S_1(n)$.

Take the constant $C_Q$ from Theorem 4.

By $S_1(n) = o(S_2(n))$, there is an $n$ so that $\forall n \geq n_0$, $C_Q \cdot S_1(n) \leq S_2(n)$.

Consider any input $x = \langle Q \rangle y$, $|y| = \lfloor n - 1 \rfloor$.

$M_s(x)$ runs $M_u_3(Q,x)$ and it stays within $S_2(1|x|) = S_2(n)$ cells.

So the run completes but it gives the opposite answer:

- If $Q$ accepts $x$, $M_s(x)$ rejects.
- If $Q$ rejects $x$, $M_s(x)$ accepts.

$Q(x)$ is not $Q(x)$, which contradicts $L(M_s) = L(Q)$.

$Q(x)$ cannot exist, so $L(M_s) \notin \text{DSPACE}(S_1(n))$.

The case for $M_t$ and time is similar where $M_t$ has a separate clock that counts up to (or down from) $t_2(n)$. We get for $n > n_0$:

\[
\frac{n+1}{t_2(n)} + \frac{t_1(n)}{t_2(n)} \leq t_2(n).
\]

So the run $M_t(x)$ with $|y| = n - 1 < S_2(n), x = \langle Q \rangle y$ finishes before $t_2(1|x|)$ steps are up, so we get the same contradiction.