Defn: A quantified Boolean formula (qbf) adds quantifiers $\exists x_i$ $\forall x_i$ to an ordinary B.F.

*note: $\phi(x_1, \ldots, x_n)$ is satisfiable iff the qbf $(\exists x_1) \cdots (\exists x_n) \phi(x_1, \ldots, x_n)$ is true.

$\forall(x_1, \ldots, x_n)$ is a tautology iff the qbf $(\forall x_1) \cdots (\forall x_n) \psi(x_1, \ldots, x_n)$ is true.

Defn: TQBF (aka QBF) is the language of all true qbf's. These can have alternating quantifiers $(\forall x_1, \ldots, x_m)(\exists y_1, \ldots, y_r)(\forall z_1, \ldots, z_l)$ etc.

Theorem: TQBF is complete for PSPACE under $\leq^P$ reduction. The proof will also show: Savitch's Theorem. For any reasonable $\text{SNP}$, $\text{NPSPACE}(\text{E} \text{LS}(n))$ is in $\text{PSPACE}(\text{E} \text{LS}(n)^2)$.
Proof: \( TQGF \in \text{DSPACE}[O(n)] \) because we can do brute-force evaluation of the qbf using only the space occupied by all the variables.

For completeness, let any \( A \in \text{DSPACE}[\text{ran}(n)] \) begin (where \( \text{ran}(n) = n^k \) for some \( k \)).

Ah, \( \text{DSPACE}[\text{ran}(n)] \)

Take an NTM \( N \) st. \( L(N) = A \) and \( N \) runs in space \( \text{ran}(n) \). To show \( A \leq_m \text{TQGF} \) we need to show

\[
x \in A \iff \psi(x) \text{ is true.}
\]

So we need to describe \( \psi \), given \( x \), and argue that \( \psi \) is computable in \( \text{logspace} \).

Key idea: Think again of \( G_x \), the ID graph with edges \( (I, J) \) st. \( I \models J \). Then \( x \in A \iff G_x \) has a path of length \( \approx O(\text{ran}(n)) \) from the start ID \( I_{0,x} \) to \( \{ \text{the accepting ID} \} \).

Put \( 2^r = 2^{O(\text{ran}(n))} \), so \( r = r(n) \).
For $0 \leq j \leq r$, define $\Psi_j(I, K)$ to hold if $I \vdash_N K$ in at most $2^j$ steps.

So: $x \in A \iff \exists r. (I_0(x), I_f)$.

$\iff (\exists J) \Psi_{r-1}(I_0(x), J) \land \Psi_{r-1}(J, I_f)$

Generally,

$\Psi_j(I, K) \iff (\exists J) \Psi_{j-1}(I, J) \land \Psi_{j-1}(J, K)$.

$\iff (\exists J) (\forall I', J'):
\left[ (I' = I \land J' = J) \lor (I' = J \land J' = K) \right] \Rightarrow \Psi_{j-1}(I', J')$.

This is a single branch recursion. At bottom

$\Psi_0(I, K) = \bot$ if $I = K$ or $I \vdash_1 K$.

Total size is $r \times |\Psi_0(I, K)| = O(r^2)$.

Easy to compute and translate to QBFs \rightarrow Wd.