Def: An oracle Turing machine (OTM) has a special query tape (write only) and query state $q'_3$. Associated to the OTM $M$ is either an oracle language $B \subseteq \Sigma^*$ or an oracle function $h: \Sigma^* \to \Sigma^*$.

Whenever $M_{B}(x)$ enters $q'_3$ with a query string $y$ on the query tape, control branches to a "yes state" $q_i$ if $y \in B$, or a "no state" $q'_i$, if $y \notin B$.

Whenever $M_{h}(x)$ enters $q'_3$, $y$ is replaced in the "magical" step by the function value $h(y)$, which $M$ can then read, until it begins to write another query.

Running times of a Def OTM $M_{B}(x)$ or computations by a non-def OTM (NDef) $M_{h}(x)$ were counted exclusively. So it seems, except...

Accordingly we define closure $OTIME_{B}^{\Sigma^*}(f(n))$, $NPTIME_{B}^{\Sigma^*}(f(n))$ relative to oracle $B(x)$ and $P^{B} = \bigcup_n OTIME_{B}^{\Sigma^*}(O(n))$, $NP^{B} = \bigcup_n NPTIME_{B}^{\Sigma^*}(O(n))$, etc.

Examples: My favorite first example involves computing $ab$ with an oracle for the squaring function $C^2$ in $O(n)$. Note that addition and other $O(n)$ algorithms can be defined as well.

Let $f: \mathbb{N} \to \mathbb{N}$ be any function we want to compute.

Define $B_{f} = \{w \in \mathbb{N} | w \leq f(w) \}$ and $B_{f}^{+} = \{w \in \mathbb{N} | w \leq f(w) \}$.

If I know that $|f(x)| = o(p(x))$ for some polynomial $p(x)$, then we can compute $f(x)$ with oracle $B_{f}$ in time $O(p(|x|))$ via binary search. Begin by querying $y = (\sqrt{x} + 1)$ or $y = \sqrt{x}$ to get $y'$.
Most important case: $f(m) = \text{the unique prime factorization of } m = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}.$

By essentially, $\exists \langle w, m \rangle$ where $m$ has a prime factor $q$ such that $w < q$.

This reduces to the solution of $\text{FIND} = \text{the least prime factor of } m = \langle w, m \rangle$.

Denote $B_f = \{ \langle w, m \rangle : w \text{ is a prefix of the unique (sub)factorization of } m \}$.

**FACT:** $B_f$ and $B_{\phi_f}$ are both in NP and coNP.

Point: To verify $\langle w, m \rangle \in B_f$, guess the unique prime factorization of $m$.\(\forall y \neq w\),

And to verify $\langle w, m \rangle \notin B_{\phi_f}$, assume $w \notin \phi_f$ or $w \notin \phi_f$.

do the same, guess, and check $w$ has\.

\[ \begin{array}{l}
\text{If } W = \phi_f \text{ or } B_f \text{ or } \phi_f \text{ or } B_f, \\
\text{We can verify the uniqueness. (AKS 2004)}
\end{array} \]