CSE681, Spring 2012 Problem Set 5 Due Mon. 4/30

Reading and Exercises:

This is the last problem set before the take-home final. Read the "Advanced" lectures through A7.

(1) Show that for any fixed prime p, $\mathsf{ACC}^0[p]$ can simulate a mod-p^2 gate. *Hint:* What can you say about the congruence of n-choose-m modulo p? How about when $m = p$ and $n = pk$ for some integer k?

Then show this for circuits of Boolean and mod-p^e gates, for $e \geq 3$. For non-prime m and $e \geq 2$, can $\mathsf{ACC}^0[m]$ circuits simulate a mod-m^e gate? (18 + 18 + 18 = 54 pts.)

(2) Finish the details of proving that $\mathsf{mSO}[\leq]$ represents exactly the class REG of regular languages. It suffices to do the forward direction, as lecture showed the main point of the converse where unrestricted Kleene star introduces a second-order existential quantifier. Use marked words in the "layered" style of lectures, and prove by induction on standard regular expressions a lemma showing that chopping off the bottom layer always simulates an existential quantifier on the corresponding variable. Note that this induction needs no case for $\cap$ or $\sim$ (i.e., complements), but the regular languages are closed under both operations—so why isn't this lemma violated by the counterexample for $\cap$ shown in lecture. (24 pts. Possible open-ended extra credit if you explore other possible ways to circumvent the $\cap$ problem, and/or whether there are cases of small $\mathsf{FO}[\leq]$ formulas that require large SF expressions.)

(3) Advanced lecture A4, exercises on page 13, 3 and 4: write formulas in $\mathsf{FO}(+,*)$ that represent the numeric predicates "y is a power of 2" and "y is a power of 2 and when x is written as a sum of powers of 2, y appears in that sum." (6+12 = 18 pts.)

(4) Suppose that R is a *-free regular expression over the "tandem alphabet" $\Sigma \times \Sigma$. Suppose that for all $x \in \Sigma^*$ there is a unique $y \in \Sigma^*$, $|y| = |x|$, such that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$ matches R. (Recall that $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] = (x_1, y_1)(x_2, y_2) \cdots (x_n, y_n)$ over $\Sigma \times \Sigma$.) Show that the function $f_R(x) = y$ is computable by DLOGTIME-uniform AC^0 circuits. (18 pts.)

(5) Same problem as (4), except that now the basis includes "dominoes" (c, ϵ) where $c \in \Sigma$ but ϵ is the empty string, and you are to classify f_R into TC^0. Thus R becomes a true "2D regular expression" over $\Sigma^* \times \Sigma^*$, and y need no longer have the same length as x. (12 pts., for 126 regular-credit points on the set. For 12 pts. extra credit, can you find an R for which f_R is complete for TC^0 under AC^0 reductions?)

(X) Same problem as (4), except that the basis also includes (ϵ, c). Now what? (I don't know the answer, and wonder if f_R is even guaranteed to be computable now...Points ad-lib.)