CSE696 Examples Prefacing HW2

Show that the class of languages that belong to DTIME[$O\left(n^{2-\epsilon}\right)$] for some $\epsilon > 0$ is recursively presentable.

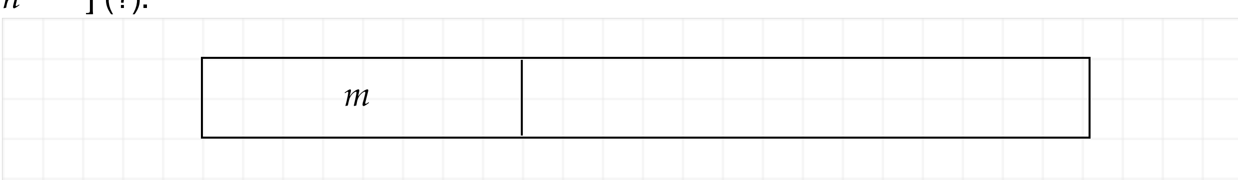
We may assume that for all rational $\epsilon > 0$, the function $n^{2-\epsilon}$ is fully time constructible, so that in $O\left(n^{2-\epsilon}\right)$ steps a TM T on any input of length n can write down the value $m = \lfloor n^{2-\epsilon} \rfloor$ on a tape, so that it can count down in real time from it.

Let us begin with a recursive presentation $[P_i]$ of all poly-time bounded machines. (We may presume that every P_i occurs infinitely often in this enumeration as $P_{i'}$ for infintiely many $i' > i$.)

Nomenclature: For fixed ϵ, our machines are $Q_{i\epsilon} =$ "P_i with the $n^{2-\epsilon}$ clock attached." What if I want to allow $kn^{2-\epsilon}$ time for any fixed constant k? What is the nomenclature now? $Q_{i,\epsilon,k}$ = "P_i with $kn^{2-\epsilon}$ clock attached." Note that each $Q_{i,\epsilon,k}$ runs in $O\left(n^{2-\epsilon}\right)$ time. For any language L that is accepted in $O\left(n^{2-\epsilon}\right)$ time, some P_i accepts L in that time, and for some k large enough to be the constant in the $O,$ when we attach the $kn^{2-\epsilon}$ clock, it does not disturb the output of P_i on any input, so that $L(Q_{i,\epsilon,k}) = L(P_i) = L$. Hence $[Q_{i,k,\epsilon}]_{i,k=1}^{\infty}$ is a recursive presentation of DTIME[$O\left(n^{2-\epsilon}\right)$] for that ϵ.

Now we want to define a recursive presentation of the class of languages that belong to DTIME[$O\left(n^{2-\epsilon}\right)$] for some $\epsilon > 0$. It is: $[Q_{i,k,1/j}]_{i,j,k=1}^{\infty}$. All we need is some (any) effectively definable sequence ϵ_j that goes to 0, so $\epsilon_j = 1/j$ is good enough. One this is understood, fine to just write $[Q_{i,k,\epsilon}]$ without the outer subscripts.

Great question: what about time $O\left(n^{2+\epsilon}\right)$ for "some" $\epsilon > 0$? Time $O\left(n^{2+o(1)}\right)$ is probably what you really meant. What it means is that there is a function $f(n) = \epsilon_n$ that $\to 0$ as $n \to \infty$ such that the language is in DTIME[$n^{2+f(n)}$]. Note: DQL is contained in DTIME[$n^{1+1/n}$] (?).



Show that $\#P$ is closed under the operations $f + g$ and $f * g$ with additive overhead in a related sense. This also serves as a work-in for the next problem.

Set-Up:

By $f, g \in \#P$, we can take predicates $R(x,y), S(x,z)$ with bounding polynomials $p(n), q(n)$ respectively such that for all x, $f(x) = \#^p y.\, R(x,y)$ and $g(x) = \#^q z.\, S(x,z)$. We need to show that $h(x) = f(x) + g(x)$ belongs to $\#P$, which means we need to find a poly-time decidable predicate $T(x,w)$ and a bounding polynomial $r(n)$ such that $h(x) = \#^r w.\, T(x,w)$.

Moreover, we need T to have small overhead, in the sense that given circuits C_n for $R(x,y)$ and D_n for $S(x,z)$, we get a circuit E_n for $T(x,w)$ with overhead [...].

Execution: Build E_n as: E_n adds an bottom OR gate connected from the outputs of C_n and D_n, giving the idea that $E(x,w) = "R(x,y)\ OR\ S(x,z)"$. But how does w relate to y and z?

$C_n(x,y)$ $\qquad\qquad\qquad\qquad$ $D_n(x,z)$

$E_n(x, \quad w = \text{.............}\ ) \equiv (w = 0y \text{ for some } y \text{ and } R(x,y)\ ) \text{ or } (w = 1z \text{ for some } z \text{ and } S(x,z)\ )$

Let Y be the set of good y's and Z the set of good z's. What set adds both together? Not $Y \times Z$ which is what we did below---that multiplies them. Not $Y \cup Z$ though that is closer, because there might be overlaps. Use the join $Y \oplus Z = \{0y : y \in Y\} \cup \{1z : z \in Z\}$.

Remaining technical niggle: what if $q(n) \neq p(n)$, so what is $r(n)$? Assuming $q(n) > p(n)$ without loss of generality, we can make the first case read $w = 0\left(0^{q(n)-p(n)}\right)y$ for some y, so that w always has length exactly $r(n) = q(n) + 1$.

Try: $E(x,w) \equiv w = yz \text{ and } R(x,y) \text{ and } S(x,z)$. What function is $\#^r w.\, E_n(x,w)$? It is $h(x) = f(x)*g(x)$ --- so we have actually solved the second problem first! For multiplication, we got $E_n(x,yz) = C_n(x,y)\ \&\&\ D_n(x,z)$, so $s(E_n) = s(C_n) + s(D_n) + 1$ counting wires. When $g = f$ this is linear: $2s + 1$. What happens to the size for addition? Still has $s(C_n) + s(D_n)$ in general because E_n needs to include both R and S. But when $R = S$, can we save...?

Verification: (can be folded in with the above)