Closed book, closed-notes, closed neighbors, 48 minutes. Please do ALL FOUR problems in the exam booklets provided. Please *show all your work*; this may help for partial credit. The exam totals 80 pts., subdivided as shown.

(1) (21 + 3 = 24 pts. total)
For all $n \geq 1$, define

$$s(n) = \sum_{r=1}^{n} \frac{1}{r^2}.$$

(a) Prove by induction that for all $n \geq 1$, $s(n) \leq \frac{2n}{n+1}$. (*Hint:* at the end, carefully cross-multiply by n^2 on one side and $(n+1)$ on the other—if you use "k to $k+1$" style you will have similar but different quantities here.)

(b) Deduce that the infinite sum

$$s(\infty) = \sum_{r=1}^{\infty} \frac{1}{r^2}$$

converges—by giving the least integer t you can such that $s(\infty) \leq t$.

(2) (3+3+6+1+1+3 = 17 points total)
For all integers n, define $f(n)$ to be the nearest multiple of 3 to n. Note that if n is a multiple of 3, then $f(n) = n$.

(a) Give the values $f(n)$ for $n = 0, 1, 2, 3, 4, 5$.

(b) What, therefore, is $f(\{0, 1, 2, 3, 4, 5\})$? What is its cardinality?

(c) What is $f^{-1}(\{3, 4, 5, 6\})$? Find a proper subset $S \subset \{3, 4, 5, 6\}$ such that $f^{-1}(S)$ gives the same answer.

(d) Is f 1-to-1?

(e) Is f onto the set of integers?

(f) Which of the following is a formula for $f(n)$ when $n \geq 0$? (Recall $\lfloor x \rfloor$ means the greatest integer y such that $y \leq x$.)

(i) $3\lfloor \frac{n+1}{3} \rfloor$

(ii) $3\lfloor \frac{n}{3} \rfloor$

(iii) $3\lfloor \frac{n+2}{3} \rfloor$

(iv) $3\lfloor n+1 \rfloor$

Exam continues overleaf

(3) (9 + 12 + 3 = 24 points total)

Let A, B, C be subsets of some universe U. Consider the proposition

$$P \equiv A \setminus (\tilde{B} \cup \tilde{C}) \subseteq B \cap C.$$

(Note: the text would write $A - (\bar{B} \cup \bar{C}) \subseteq B \cap C$ instead.)

(a) Write the corresponding logical proposition, using just a for "$x \in A$", and b similarly for B, c similarly for C. Call it ρ (Greek rho).

(b) Prove that ρ is a tautology. Any covered proof method, e.g. truth-tables or some other "semantic proof," proof rules or some other "syntactic proof," is AOK.

(c) Deduce that P itself is always true.

(4) (5 × 3 = 15 pts.)

True/False. Please write out the words `true` and `false` in full. Brief justifications are not needed, but might help for partial credit.

(a) The power set of the empty set is the empty set.

(b) The power set of the empty set has cardinality $2^0 = 1$.

(c) If $P(0)$ is true, $P(1)$ is true, and for all $n \geq 2$, $P(n-2) \longrightarrow P(n)$ is true, then $(\forall n)P(n)$ is true, where n ranges over the domain of natural numbers.

(d) The complement of the union of two sets is always a subset of one of the sets.

(e) The difference of two sets is always a subset of their intersection.

End of Exam