High School Trig Is Powerful

How to cheat a decision problem

Leopold Kronecker proved many great theorems including one on Diophantine Approximation theory. This is the theory of how closely certain real numbers r can be approximated by rational numbers a/b. How many correct decimal places of r can you get, in terms of the number of digits in the integers a and b? This is an early kind of complexity theory.

Today I want to talk about a decision question that arose in reading a paper on number theory. It involves a simple question about equations involving cosine functions, yet seems hard to resolve.

Let state my feelings about such functions: I love and hate trigonometry. Let me explain why I love it, then we can move on to why I hate it. Okay?

One of the great things about trig functions is that the play a major role in the analysis of some important number theory problems. The Prime Number Theorem, the statement that the number of primes less than x is approximately $x/\ln x$ is a deep theorem. We now have many proofs of this key fact about the behavior of the primes. Some use complex analysis, some are elementary, some use a combination.

But one of the early proofs used a simple fact from trigonometric equations. They used the identity

$$3 + 4\cos(x) + \cos(2x) = 2(1 + \cos(x))^2.$$

It maybe seem amazing that such a deep theorem can be based on such a simple trig result, yet it is true. The reason it is so useful is that the identity implies the *inequality*,

$$3 + 4\cos(x) + \cos(2x) = 2(1 + \cos(x))^2 \geq 0.$$

The simple connection is that the real part of $1/n^s$ can be expressed using cosine functions—see here for more details.

Schooling Me on Equations

I can imagine that I was once asked to prove the above identity back in high school—it is pretty basic. Speaking of high school that is why I hate trig. When I took it in school, it seemed like it was a lot of memorization—what is the definition of all those strangely named functions? I also learned trig in summer school from a Mr. Ditka—I do not know his first name. He was like his name—tough—and later when I saw the famous football coach Mike Ditka on television I wondered if they might have been related.

Mr. Ditka, I can call him nothing else, taught me an important lesson about math. During that summer class I remember getting a problem wrong on an exam. I walked up to him after class with my answer and asked why I got zero points. The question was about writing down the equation for something, and the answer I had written was

$$x^2 - x + 1.$$

I asked why did I get zero points, since this was the correct expression. Mr. Ditka looked at me and said,

> I asked for an **equation**. You did not write an equation. Zero points.

He was right, if I only had added the equality sign and written

$$x^2 - x + 1 = 0$$

I would have score full points. Oh well. Lesson learned.

Not The Question

I once tried to see if there were more general equations like the cosine one above. I thought about this question: Suppose,

$$a_1 \cos(b_1 x) + \cdots + a_m \cos(b_m x) \geq 0,$$

where the $a_i > 0$'s and $b_i > 0$'s. Is there an inequality like this?

The answer is that there is no such inequality: note that any correct inequality must have a constant term. So the claim is that without a constant term there is no such inequality. I will prove it in a moment.

The Question

The real question concerns determining the possible values of a sum of cosine functions. This is a natural problem, but is also an important problem in number theory. Define a function f by

$$f(t) = \sum_{k=1}^{m} a_k \cos(t \ln k).$$

We want to determine whether or not this function can be equal, even approximately, to some given value. Since the a_k are fixed, it is a question in **one** variable—how had can that be? The initial idea might be to just use calculus to solve the problem. But there is a problem. The function is defined for all real t, and therefore the answer can be any real number. Since this is an infinite interval, it seems to me, that no standard search method can work. If we could bound the value of t then we could solve the problem.

The key insight is to use Diophantine approximation theory. Let's look at a simple example first. Let $f(t)$ be

$$a\cos(t\ln 2) + b\cos(t\ln 3) + c\cos(t\ln 4) + d\cos(t\ln 5).$$

This is equal to

$$a\cos(t\ln 2) + c\cos(2t\ln 2) + b\cos(t\ln 3) + d\cos(t\ln 5).$$

By approximation theory this can be as close to

$$a\cos(\theta) + c\cos(2\theta) + bx + dy,$$

where θ is arbitrary and x, y are both at most 1 in absolute value. Thus, we need to study the minimum of

$$a\cos(\theta) + b\cos^2(\theta) - b + cx + dy.$$

This can be solved easily and bounded.

The reason for the latter claim is the following theoem:

Theorem 0.1. *Let $1, \alpha_1, \ldots, \alpha_n$ be real numbers that are linearly independent over the rationals. Let $\nu_1, \ldots, \nu_n$ be arbitrary real numbers, $\epsilon > 0$ and $N > 0$ an integer. Then, there exist integers $p_1, \ldots, p_n$ and an integer $q > N$ so that for all i,*

$$|q\alpha_i - p_i - \nu_i| < \epsilon.$$

In general the cost is based on the number of independent values that are needed.

Not The Question—The Proof

Suppose for all x,

$$f(x) = a_1\cos(b_1 x) + \cdots + a_m\cos(b_m x) \geq 0,$$

where the a_i's and b_i's are non-zero reals.

The key is to look at the integral $\int_0^t f(x)dx$. This flips the cosine functions to sine functions, and yields

$$a_1' \sin(b_1 x) + \cdots + a_m' \sin(b_m x),$$

which still holds for all x. But at $x = 0$ it is 0, and for x just below 0 the expression becomes negative, a contradiction.

Open Problems