CSE 431/531: Analysis of Algorithms

Approximation and Randomized Algorithms

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Department of Computer Science and Engineering
University at Buffalo

Outline

1. Approximation Algorithms
2. Approximation Algorithms for Traveling Salesman Problem
3. 2-Approximation Algorithm for Vertex Cover
4. $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
5. Randomized Quicksort
    - Recap of Quicksort
    - Randomized Quicksort Algorithm
6. 2-Approximation Algorithm for (Weighted) Vertex Cover Via Linear Programming
    - Linear Programming
    - 2-Approximation for Weighted Vertex Cover

Approximation Algorithms

An algorithm for an optimization problem is an α-approximation algorithm, if it runs in polynomial time, and for any instance to the problem, it outputs a solution whose cost (or value) is within an α-factor of the cost (or value) of the optimum solution.

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  - $\alpha \geq 1$ and we require sol $\leq \alpha \cdot$ opt
- For maximization problems, there are two conventions:
  - $\alpha \leq 1$ and we require sol $\geq \alpha \cdot$ opt
  - $\alpha \geq 1$ and we require sol $\geq$ opt$/\alpha$

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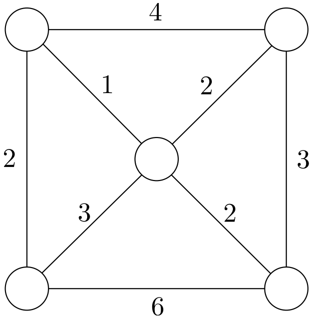
Recall: Traveling Salesman Problem

- A salesman needs to visit n cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city 1
- Goal: find a tour with the minimum cost



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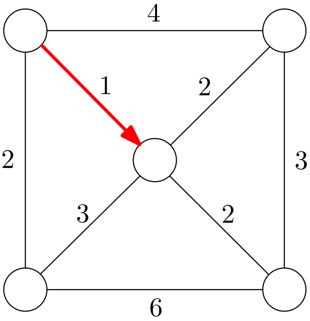

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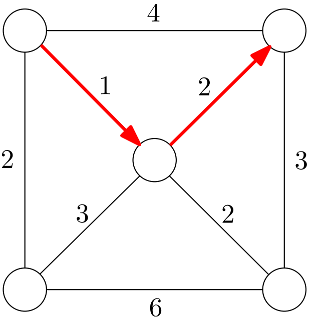

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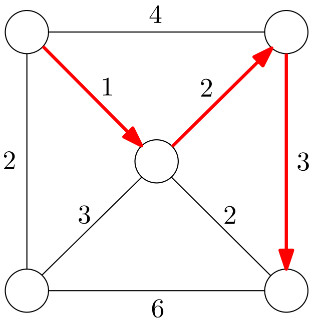

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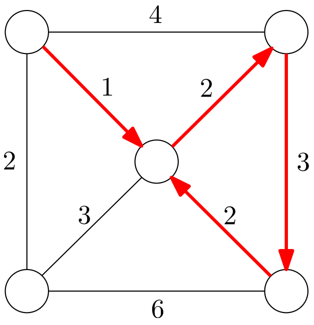

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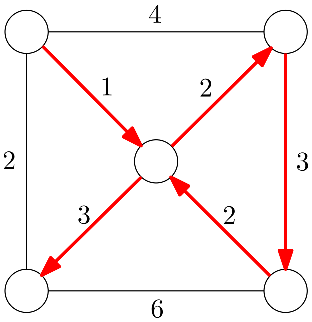

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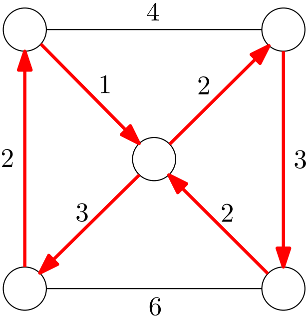

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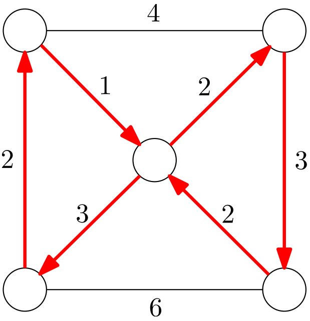

Travelling Salesman Problem (TSP)

Input: a graph $G = (V, E)$, weights $w : E \to \mathbb{R}_{\geq 0}$

Output: a traveling-salesman tour with the minimum cost

2-Approximation Algorithm for TSP

TSP1(G, w)

1. $MST \leftarrow$ the minimum spanning tree of G w.r.t weights w, returned by either Kruskal's algorithm or Prim's algorithm.
2. Output tour formed by making two copies of each edge in MST.

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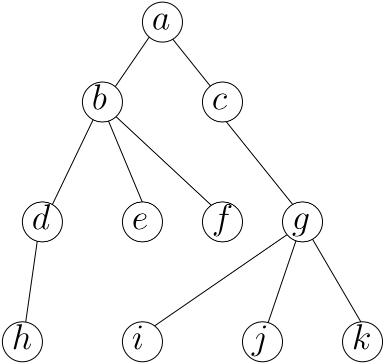

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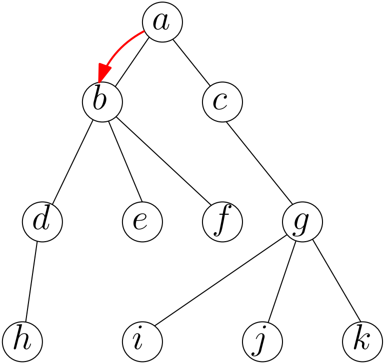

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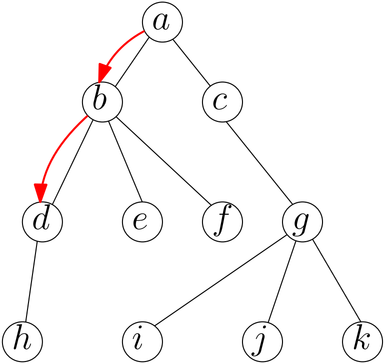

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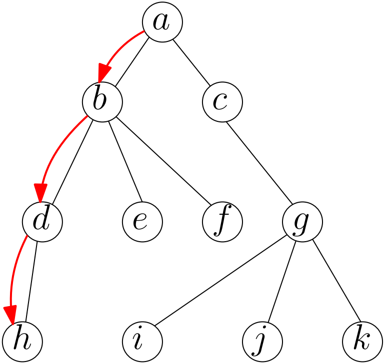

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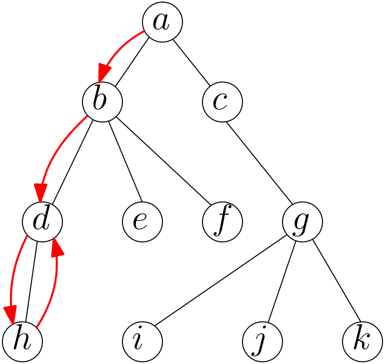

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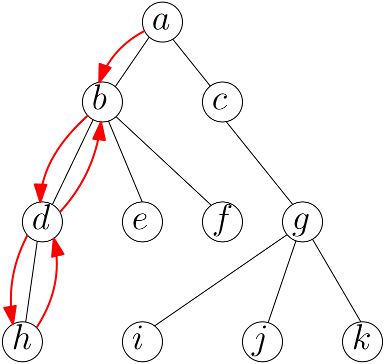

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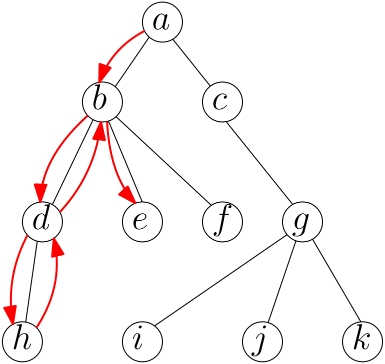

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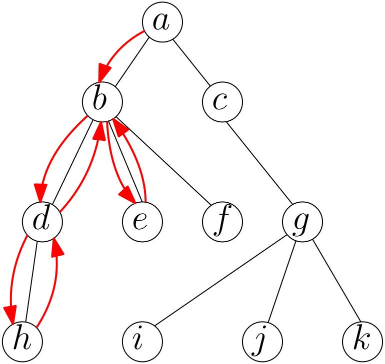

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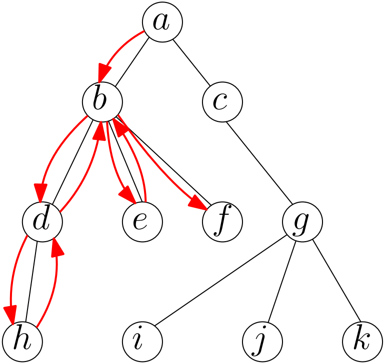

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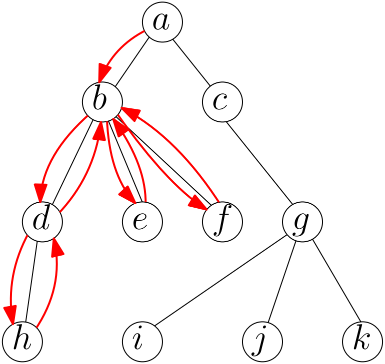

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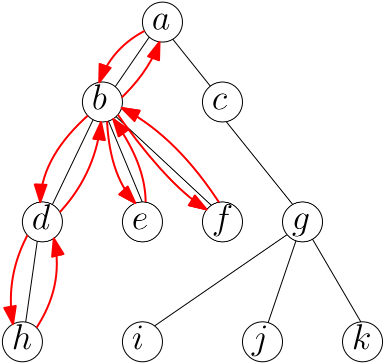

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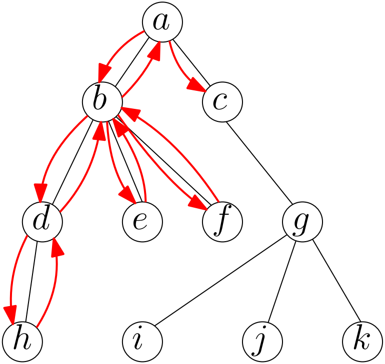

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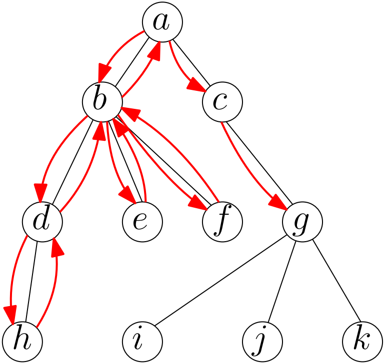

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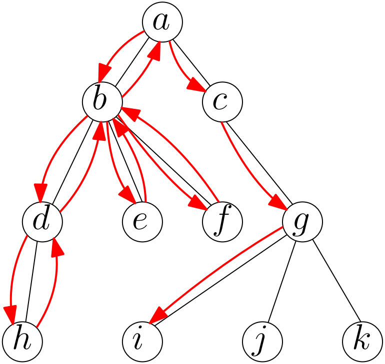

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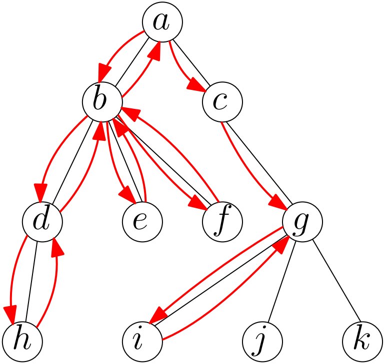

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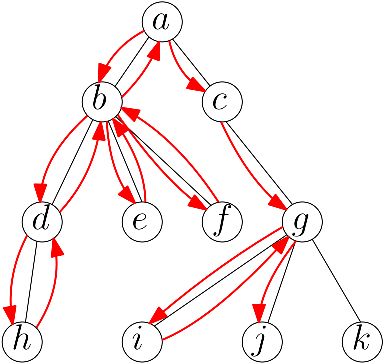

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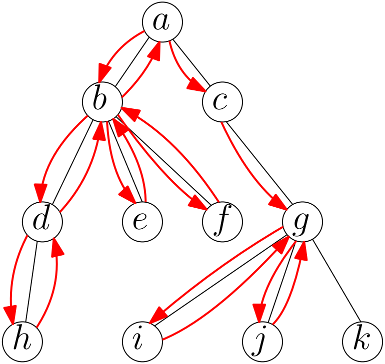

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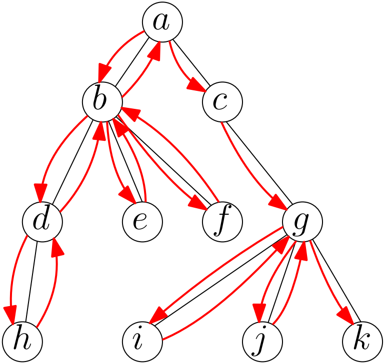

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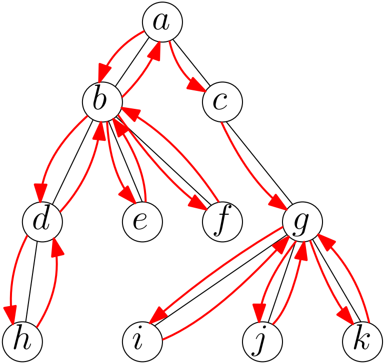

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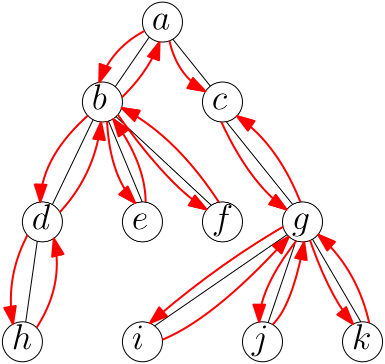

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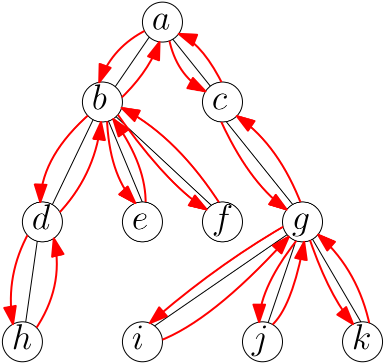

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- sol $=$ cost of tour given by algorithm TSP1
- sol $= 2 \cdot$ mst $\leq 2 \cdot$ tsp. □

1.5-Approximation for TSP

Def. Given $G = (V, E)$, a set $U \subseteq V$ of even number of vertices in V, a matching M over U in G is a set of $|U|/2$ paths in G, such that every vertex in U is one end point of some path.

Def. The cost of the matching M, denoted as cost(M) is the total cost of all edges in the $|U|/2$ paths (counting multiplicities).

Theorem Given $G = (V, E)$, a set $U \subseteq V$ of even number of verticies, the minimum cost matching over U in G can be found in polynomial time.

1.5-Approximation for TSP

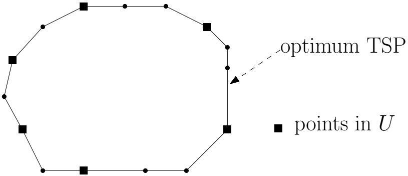
Lemma Let T be a spanning tree of $G = (V, E)$; let U be the set of odd-degree vertices in MST ($|U|$ must be even, why?). Let M be a matching over U, then, $T \uplus M$ gives a traveling salesman's tour.

Proof.

Every vertex in $T \uplus M$ has even degree and $T \uplus M$ is connected (since it contains the spanning tree). Thus $T \uplus M$ is an Eulerian graph and we can find a tour that visits every edge in $T \uplus M$ exactly once. □

1.5-Approximation for TSP

Lemma Let U be a set of even number of vertices in G. Then the cost of the cheapest matching over U in G is at most $\frac{1}{2}$tsp.



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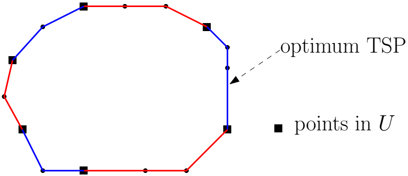

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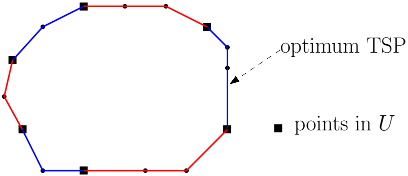

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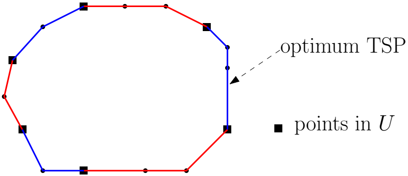

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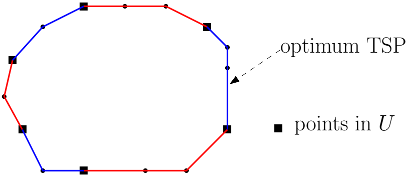

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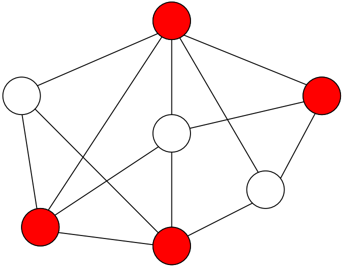
Vertex Cover Problem

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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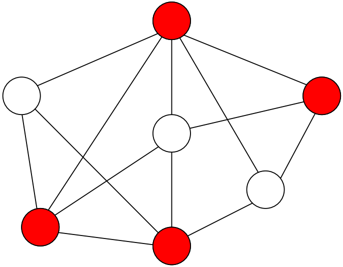

Vertex-Cover Problem

Input: $G = (V, E)$

Output: a vertex cover S with minimum $|S|$

First Try: Greedy Algorithm

Greedy Algorithm for Vertex-Cover

```
$E' \leftarrow E, S \leftarrow \emptyset$
while $E' \neq \emptyset$
    let $v$ be the vertex of the maximum degree in $(V, E')$
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Theorem Greedy algorithm is an $O(\lg n)$-approximation for vertex-cover.

- We are not going to prove the theorem
- Instead, we show that the $O(\lg n)$-approximation ratio is tight for the algorithm

Bad Example for Greedy Algorithm

- L: n' vertices

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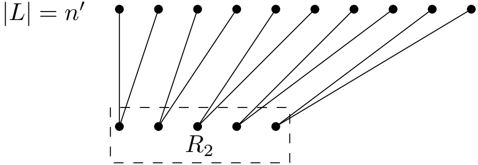

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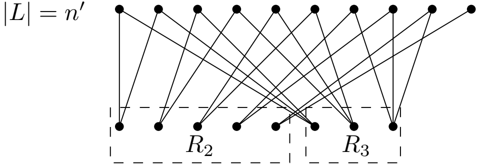

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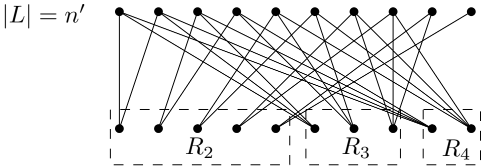

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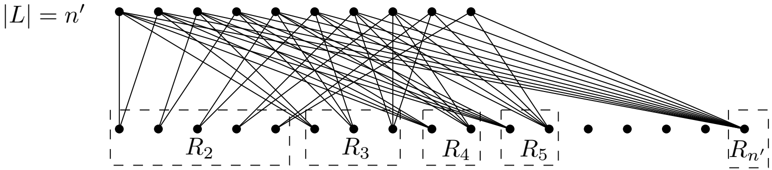

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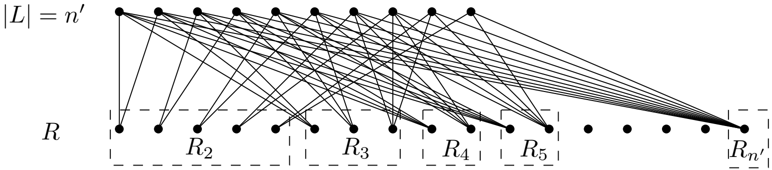

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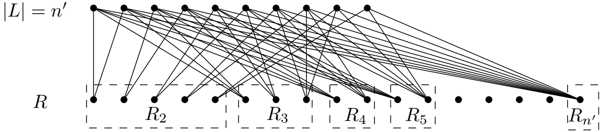

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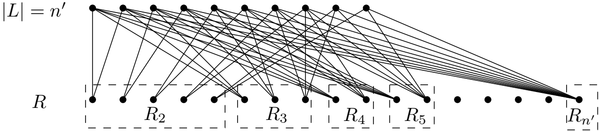

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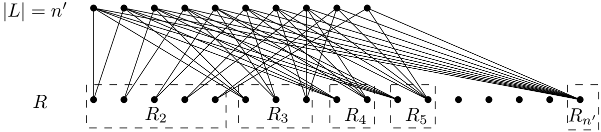

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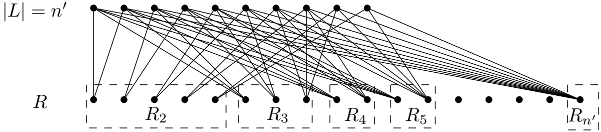

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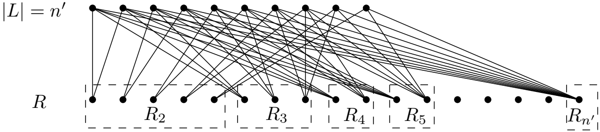

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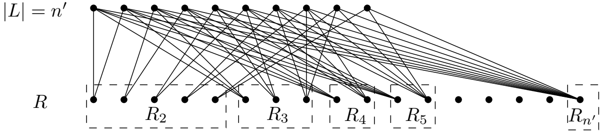

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- where $H(n') = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n'} = \Theta(\lg n')$ is the n'-th number in the harmonic sequence.

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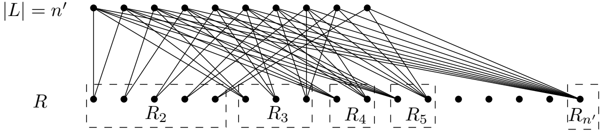

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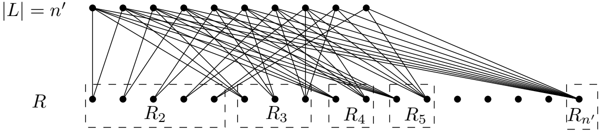

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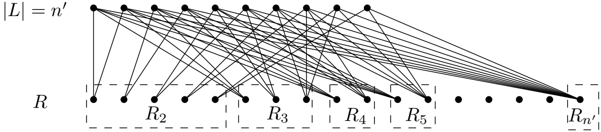

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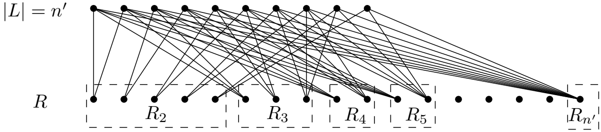

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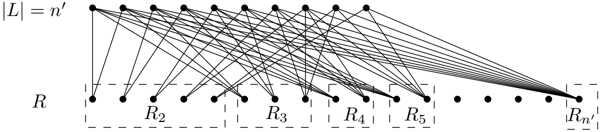

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- $\frac{|R|}{|L|} = \frac{\Omega(n' \lg n')}{n'} = \Omega(\lg n') = \Omega(\lg n)$.
- Thus, greedy algorithm does not do better than $O(\lg n)$.

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- However, the approximation ratio is not so good
- We now give a somewhat “counter-intuitive” algorithm,
- for which we can prove a 2-approximation ratio.

2-Approximation Algorithm for Vertex Cover

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while $E' \neq \emptyset$
    let $(u, v)$ be any edge in $E'$
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- The counter-intuitive part: adding both u and v to S seems to be wasteful
- Intuition for the 2-approximation ratio: the optimum solution must cover the edge (u, v), using either u or v. If we select both, we are always ahead of the optimum solution. The approximation factor we lost is at most 2.

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Theorem The algorithm is a 2-approximation algorithm for vertex-cover.

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2. Approximation Algorithms for Traveling Salesman Problem
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4. $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
5. Randomized Quicksort
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Max 3-SAT

Input: n boolean variables $x_1, x_2, \cdots, x_n$

m clauses, each clause is a disjunction of 3 literals from 3 distinct variables

Output: an assignment so as to satisfy as many clauses as possible

Example:

- clauses: $x_2 \vee \neg x_3 \vee \neg x_4, \quad x_2 \vee x_3 \vee \neg x_4,$ $\neg x_1 \vee x_2 \vee x_4, \quad x_1 \vee \neg x_2 \vee x_3, \quad \neg x_1 \vee \neg x_2 \vee \neg x_4$
- We can satisfy all the 5 clauses: $x = (1, 1, 1, 0, 1)$

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- $\mathbb{E}[Z] = \mathbb{E}\left[\sum_{j=1}^{m} Z_j\right] = \sum_{j=1}^{m} \mathbb{E}[Z_j] = \sum_{j=1}^{m} \frac{7}{8} = \frac{7}{8}m$, by linearity of expectation. $\square$

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Theorem ([Hastad 97]) Unless P $=$ NP, there is no ρ-approximation algorithm for MAX-3-SAT for any $\rho > 7/8$.

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Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85

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quicksort(A, n)

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if $n \leq 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$        \\ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$     \\ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L.$size$)$                  \\ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R.$size$)$                  \\ Conquer
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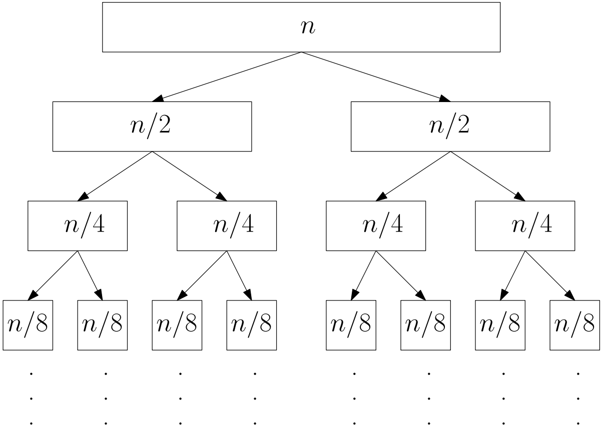

- Each level has total running time $O(n)$
- Number of levels $= O(\lg n)$
- Total running time $= O(n \lg n)$

Quicksort Can Be Implemented as an "In-Place" Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses "small" extra space.

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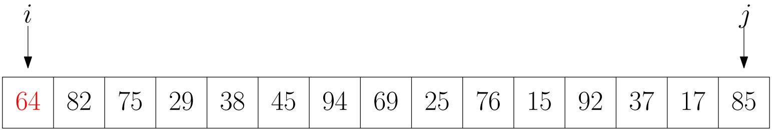

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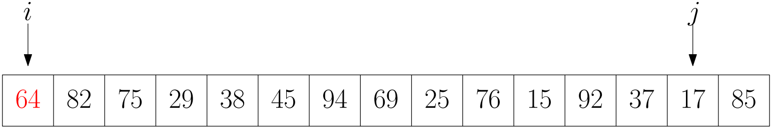

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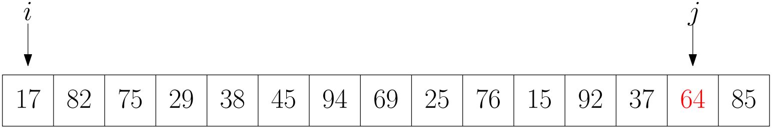

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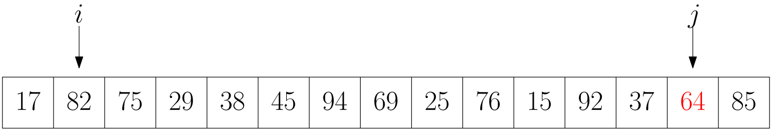

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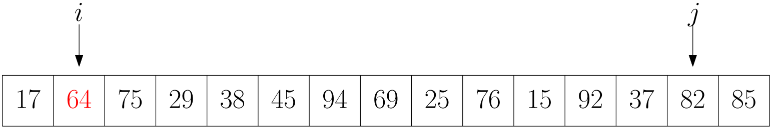

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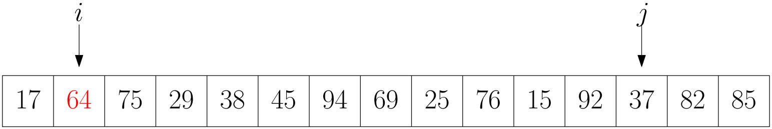

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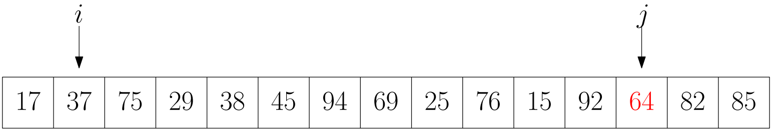

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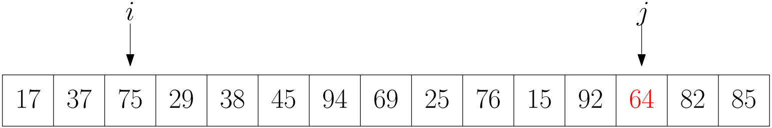

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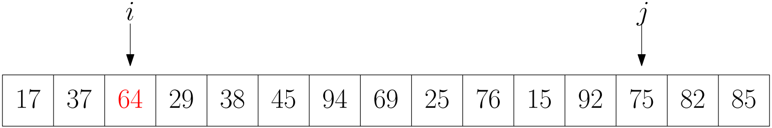

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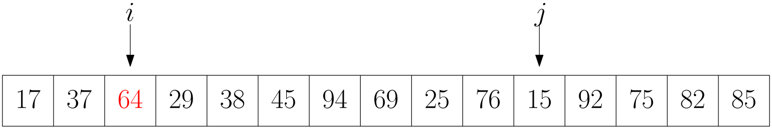

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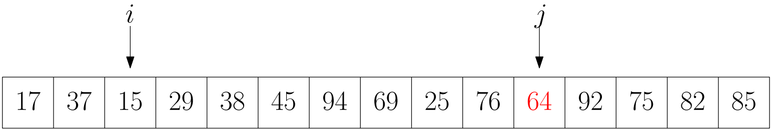

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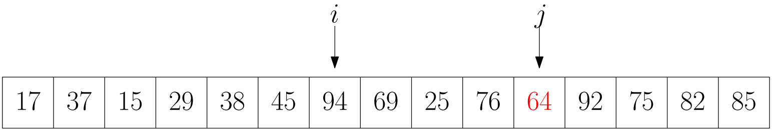

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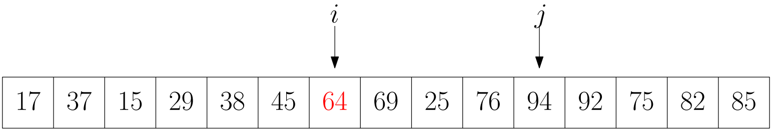

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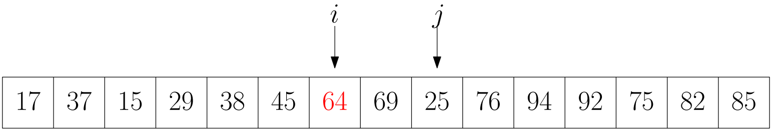

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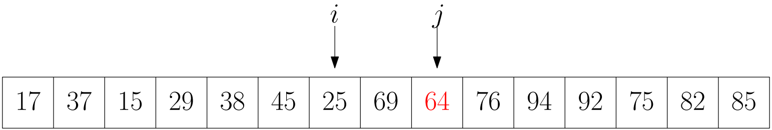

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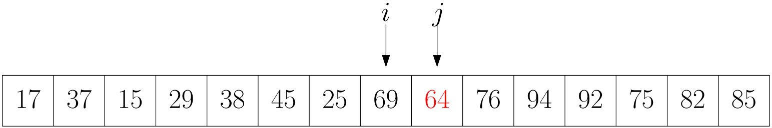

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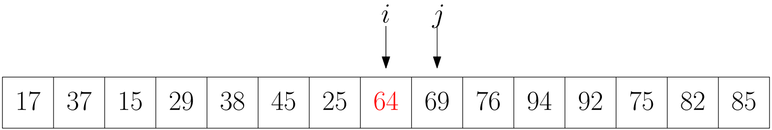

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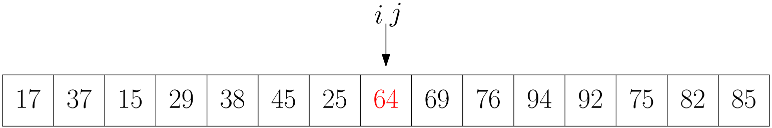

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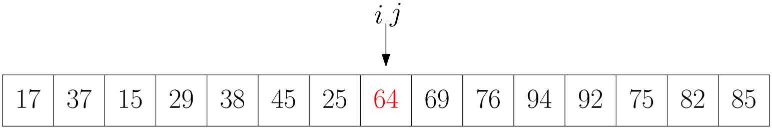

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- To partition the array into two parts, we only need $O(1)$ extra space.

Outline

Randomized Quicksort Algorithm

```
quicksort($A, n$)
    if $n \le 1$ then return $A$
    $x \leftarrow$ a random element of $A$ ($x$ is called a pivot)
    $A_L \leftarrow$ elements in $A$ that are less than $x$          \\ Divide
    $A_R \leftarrow$ elements in $A$ that are greater than $x$       \\ Divide
    $B_L \leftarrow$ quicksort($A_L$, $A_L$.size)                    \\ Conquer
    $B_R \leftarrow$ quicksort($A_R$, $A_R$.size)                    \\ Conquer
    $t \leftarrow$ number of times $x$ appear $A$
    return the array obtained by concatenating $B_L$, the array
    containing $t$ copies of $x$, and $B_R$
```

Variant of Randomized Quicksort Algorithm

```
quicksort($A, n$)
if $n \leq 1$ then return $A$
repeat
    $x \leftarrow$ a random element of $A$ ($x$ is called a pivot)
    $A_L \leftarrow$ elements in $A$ that are less than $x$          \\ Divide
    $A_R \leftarrow$ elements in $A$ that are greater than $x$       \\ Divide
unitl $A_L$.size $\leq 3n/4$ and $A_R$.size $\leq 3n/4$
$B_L \leftarrow$ quicksort($A_L$, $A_L$.size)                        \\ Conquer
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$t \leftarrow$ number of times $x$ appear $A$
return the array obtained by concatenating $B_L$, the array
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```

Analysis of Variant

3. $x \leftarrow$ a random element of A
4. $A_L \leftarrow$ elements in A that are less than x
5. $A_R \leftarrow$ elements in A that are greater than x

Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

Analysis of Variant

```
3  $x \leftarrow$ a random element of $A$
4  $A_L \leftarrow$ elements in $A$ that are less than $x$
5  $A_R \leftarrow$ elements in $A$ that are greater than $x$
```

Q: What is the probability that A_L.size $\leq 3n/4$ and A_R.size $\leq 3n/4$?

A: At least $1/2$

Analysis of Variant

```
repeat
    $x \leftarrow$ a random element of $A$
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Q: What is the expected number of iterations the above procedure takes?

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unitl $A_L$.size $\leq 3n/4$ and $A_R$.size $\leq 3n/4$
```

Q: What is the expected number of iterations the above procedure takes?

A: At most 2

- Suppose an experiment succeeds with probability $p \in (0, 1]$, independent of all previous experiments.

```
repeat
    run an experiment
until the experiment succeeds
```

Lemma The expected number of experiments we run in the above procedure is $1/p$.

Fact For $q \in (0, 1)$, we have $\sum_{i=0}^{\infty} q^i = \frac{1}{1-q}$.

Lemma The expected number of experiments we run in the above procedure is $1/p$.

Proof

$$\begin{aligned}
\text{Expectation} &= p + (1-p)p \times 2 + (1-p)^2 p \times 3 + (1-p)^3 p \times 4 \\
&\quad + \cdots \\
&= p\sum_{i=1}^{\infty}(1-p)^{i-1} i \quad = \quad p\sum_{j=1}^{\infty}\sum_{i=j}^{\infty}(1-p)^{i-1} \\
&= p\sum_{j=1}^{\infty}(1-p)^{j-1}\frac{1}{1-(1-p)} \quad = \quad \sum_{j=1}^{\infty}(1-p)^{j-1} \\
&= (1-p)^0 \frac{1}{1-(1-p)} = 1/p
\end{aligned}$$

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return the array obtained by concatenating $B_L$, the array
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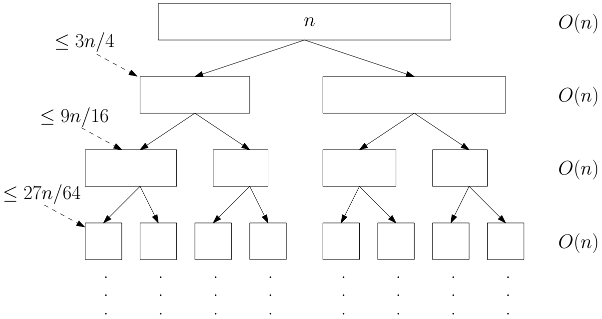
Analysis of Variant

- Divide and Combine: takes $O(n)$ time
- Conquer: break an array of size n into two arrays, each has size at most $3n/4$. Recursively sort the 2 sub-arrays.



- Number of levels $\le \lg_{4/3} n = O(\lg n)$

Randomized Quicksort Algorithm

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- Intuition: the quicksort algorithm should be better than the variant.

Analysis of Randomized Quicksort Algorithm

- $T(n)$: an upper bound on the expected running time of the randomized quicksort algorithm on n elements

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- Thus, the expected running time in this case is $\big(T(i-1)+T(n-i)\big)+O(n)$

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$$T(n) = \frac{1}{n}\sum_{i=1}^{n}\big(T(i-1)+T(n-i)\big)+O(n)$$

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$$\begin{aligned} T(n) &= \frac{1}{n}\sum_{i=1}^{n}\big(T(i-1) + T(n-i)\big) + O(n) \\ &= \frac{2}{n}\sum_{i=0}^{n-1} T(i) + O(n) \end{aligned}$$

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- Can prove $T(n) \leq c(n\lg n)$ for some constant c by reduction

Analysis of Randomized Quicksort Algorithm

The induction step of the proof:

$$
\begin{aligned}
T(n) &\le \frac{2}{n}\sum_{i=0}^{n-1} T(i) + c'n \le \frac{2}{n}\sum_{i=0}^{n-1} ci\lg i + c'n \\
&\le \frac{2c}{n}\left(\sum_{i=0}^{\lfloor n/2\rfloor-1} i\lg\frac{n}{2} + \sum_{i=\lfloor n/2\rfloor}^{n-1} i\lg n\right) + c'n \\
&\le \frac{2c}{n}\left(\frac{n^2}{8}\lg\frac{n}{2} + \frac{3n^2}{8}\lg n\right) + c'n \\
&= c\left(\frac{n}{4}\lg n - \frac{n}{4} + \frac{3n}{4}\lg n\right) + c'n \\
&= cn\lg n - \frac{cn}{4} + c'n \le cn\lg n \qquad \text{if } c \ge 4c'
\end{aligned}
$$

Exercise: Coupon Collector

Coupon Collector

Each box of cereal contains a coupon. There are n different types of coupons. Assuming all boxes are equally likely to contain each coupon, in expectation, how many boxes before you have all coupon types?

- Break into n stages $1, 2, 3, \cdots, n$
- Stage i terminates when we have collected i coupon types
- X_i: number of coupons collected in stage i
- $X = \sum_{i=1}^{n} X_i$: total number of coupons collected

Exercise: Coupon Collector

- X_i: number of coupons collected in stage i
- $X = \sum_{i=1}^{n} X_i$: total number of coupons collected
- In stage i: with probability $\frac{n-(i-1)}{n}$, a random coupon has type different from the $i-1$ types already seen
- Thus, $\mathbb{E}[X_i] = \frac{n}{n-(i-1)}$.
- By linearity of expectation:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \frac{n}{n-(i-1)} = \sum_{i=1}^{n} \frac{n}{i} = nH(n),$$

  where $H(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \Theta(\lg n)$ is called the n-th Harmonic number.
- $\mathbb{E}[X] = \Theta(n \lg n)$.

Outline

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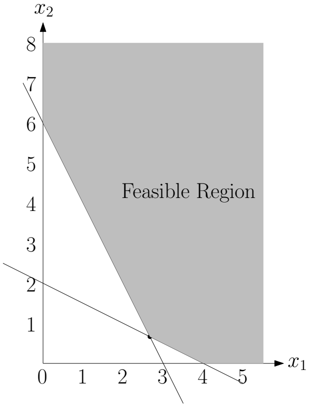
1. Approximation Algorithms
2. Approximation Algorithms for Traveling Salesman Problem
3. 2-Approximation Algorithm for Vertex Cover
4. $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
5. Randomized Quicksort
    - Recap of Quicksort
    - Randomized Quicksort Algorithm
6. 2-Approximation Algorithm for (Weighted) Vertex Cover Via Linear Programming
    - Linear Programming
    - 2-Approximation for Weighted Vertex Cover

Example of Linear Programming

$$\begin{aligned} \min \quad & 4x_1 + 5x_2 \quad \text{s.t.} \\ & 2x_1 + x_2 \geq 6 \\ & x_1 + 2x_2 \geq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$



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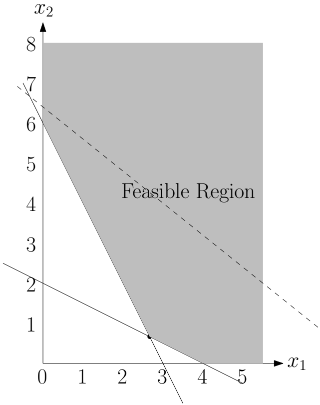

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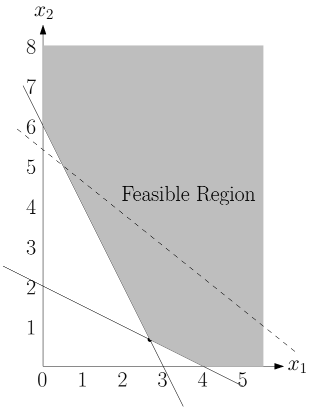

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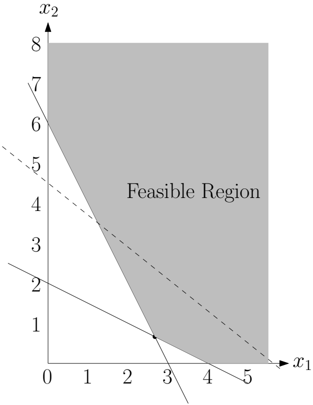

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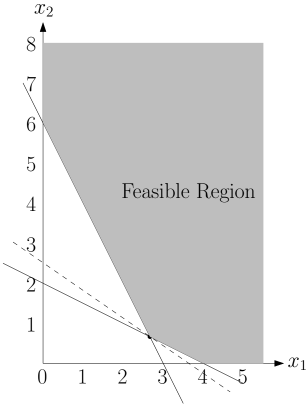

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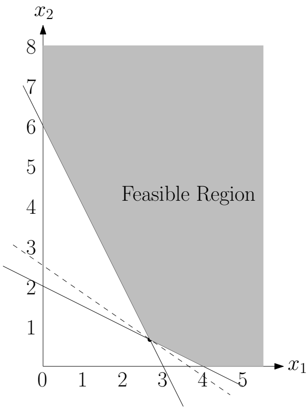
- optimum point: $x_1 = \frac{8}{3}, x_2 = \frac{2}{3}$



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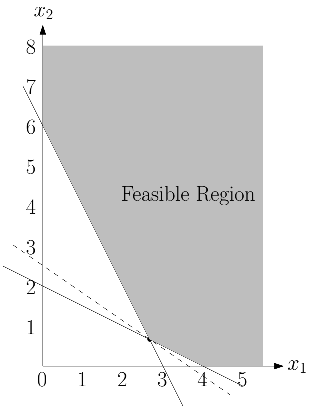
- optimum point: $x_1 = \frac{8}{3}, x_2 = \frac{2}{3}$
- value $= 4 \times \frac{8}{3} + 5 \times \frac{2}{3} = 14$



Standard Form of Linear Programming

$$\begin{aligned}
\min \qquad c_1x_1 + c_2x_2 + \cdots + c_nx_n \qquad & \text{s.t.} \\
\sum A_{1,1}x_1 + A_{1,2}x_2 + \cdots + A_{1,n}x_n &\geq b_1 \\
\sum A_{2,1}x_1 + A_{2,2}x_2 + \cdots + A_{2,n}x_n &\geq b_2 \\
\vdots \qquad \vdots \qquad \vdots \qquad \vdots \\
\sum A_{m,1}x_1 + A_{m,2}x_2 + \cdots + A_{m,n}x_n &\geq b_m \\
x_1, x_2, \cdots, x_n &\geq 0
\end{aligned}$$

Standard Form of Linear Programming

Let $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$,

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & A_{m,2} & \cdots & A_{m,n} \end{pmatrix}, \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Then, LP becomes $\min \quad c^{\mathrm{T}} x$ s.t.

$$Ax \geq b$$

$$x \geq 0$$

- $\geq$ means coordinate-wise greater than or equal to

- Linear programmings can be solved in polynomial time

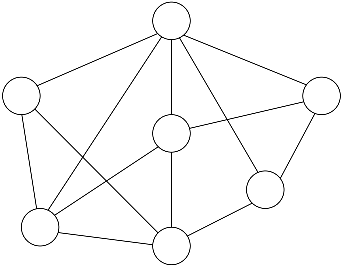
Algorithms for Solving LPs

- Simplex method: exponential time in theory, but works well in practice
- Ellipsoid method: polynomial time in theory, but slow in practice
- Internal point method: polynomial time in theory, works well in practice

Outline

1. Approximation Algorithms
2. Approximation Algorithms for Traveling Salesman Problem
3. 2-Approximation Algorithm for Vertex Cover
4. $\frac{7}{8}$-Approximation Algorithm for Max 3-SAT
5. Randomized Quicksort
   - Recap of Quicksort
   - Randomized Quicksort Algorithm
6. 2-Approximation Algorithm for (Weighted) Vertex Cover Via Linear Programming
   - Linear Programming
   - 2-Approximation for Weighted Vertex Cover

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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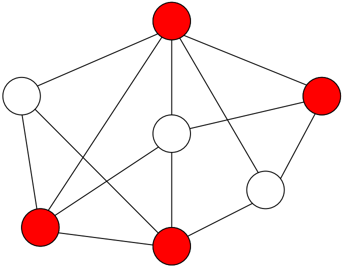

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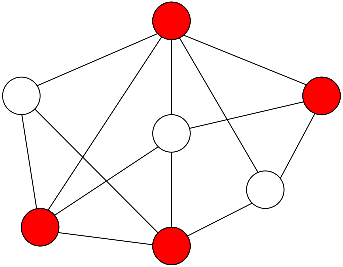

Weighted Vertex-Cover Problem

Input: $G = (V, E)$ with vertex weights $\{w_v\}_{v \in V}$

Output: a vertex cover S with minimum $\sum_{v \in S} w_v$

Integer Programming for Weighted Vertex Cover

- For every $v \in V$, let $x_v \in \{0,1\}$ indicate whether we select v in the vertex cover S
- The integer programming for weighted vertex cover:

$$
\begin{aligned}
(\text{IP}_{\text{WVC}}) \qquad \min \qquad & \sum_{v \in V} w_v x_v \qquad \text{s.t.} \\
x_u + x_v \geq 1 \qquad & \forall (u, v) \in E \\
x_v \in \{0, 1\} \qquad & \forall v \in V
\end{aligned}
$$

- $(\text{IP}_{\text{WVC}}) \Leftrightarrow$ weighted vertex cover
- Thus it is NP-hard to solve integer programmings in general

- Integer programming for WVC:

$$\begin{aligned} (\mathsf{IP_{WVC}}) \qquad \min \qquad & \sum_{v \in V} w_v x_v \qquad \text{s.t.} \\ x_u + x_v \geq 1 \qquad & \qquad \forall (u,v) \in E \\ x_v \in \{0,1\} \qquad & \qquad \forall v \in V \end{aligned}$$

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$$(\text{IP}_{\text{WVC}}) \qquad \min \quad \sum_{v\in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
$$x_v \in \{0,1\} \qquad \forall v \in V$$

- Linear programming relaxation for WVC:

$$(\text{LP}_{\text{WVC}}) \qquad \min \quad \sum_{v\in V} w_v x_v \qquad \text{s.t.}$$
$$x_u + x_v \geq 1 \qquad \forall (u,v) \in E$$
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- let IP $=$ value of (IP_{WVC}), LP $=$ value of (LP_{WVC})

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- let IP $=$ value of (IP_{WVC}), LP $=$ value of (LP_{WVC})
- Then, LP $\leq$ IP

Algorithm for Weighted Vertex Cover

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1. Solving (LP_{WVC}) to obtain a solution $\{x_u^*\}_{u \in V}$
2.
3.

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- Thus, either $u \in S$ or $v \in S$. □

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Proof.

$$\begin{aligned} \text{cost}(S) &= \sum_{u \in S} w_u \leq \sum_{u \in S} w_u \cdot 2x_u^* = 2 \sum_{u \in S} w_u \cdot x_u^* \\ &\leq 2 \sum_{u \in V} w_u \cdot x_u^* = 2 \cdot \text{LP}. \end{aligned}$$

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Proof.

$\text{cost}(S) \leq 2 \cdot \text{LP} \leq 2 \cdot \text{IP} = 2 \cdot \text{cost}(\text{best vertex cover})$. ☐