CSE 431/531: Analysis of Algorithms
Graph Basics

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University at Buffalo

Outline

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

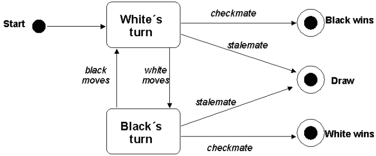
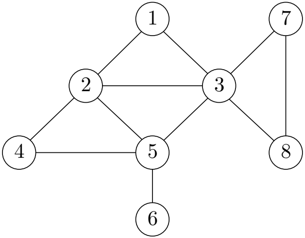


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



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- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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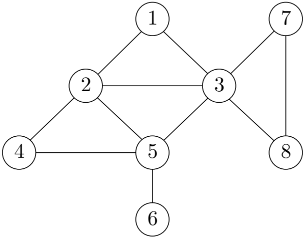

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  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
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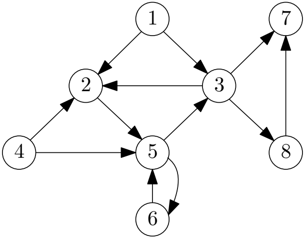
  - $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,5\}, \{3,7\}, \{3,8\}, \{4,5\}, \{5,6\}, \{7,8\}\}$

Directed Graph $G = (V, E)$



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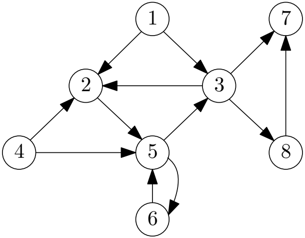

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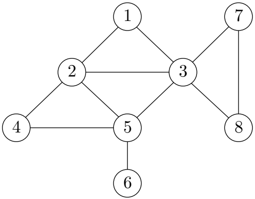
Abuse of Notations

- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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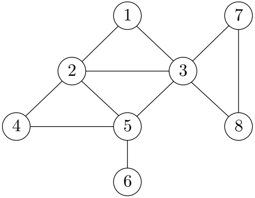
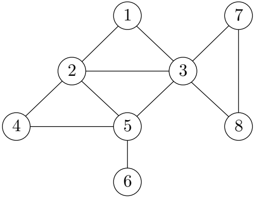
- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

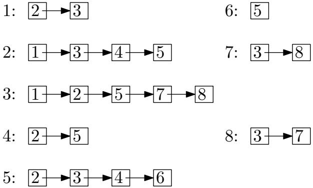


	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

Representation of Graphs





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  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n-1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u,v) \in E$		
time to list all neighbours of v		

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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u,v) \in E$	$O(1)$	
time to list all neighbours of v		

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time to list all neighbours of v	$O(n)$	

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time to list all neighbours of v	$O(n)$	$O(d_v)$

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Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

Output: whether there is a path connecting s to t in G

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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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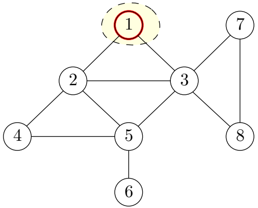

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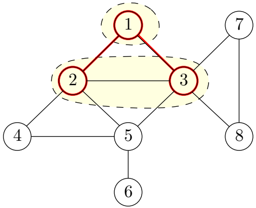

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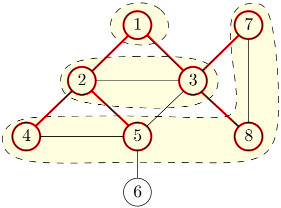

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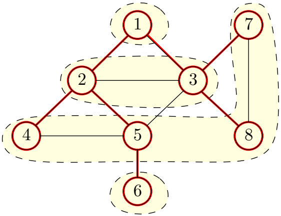


Implementing BFS using a Queue

BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$

mark s as "visited" and all other vertices as "unvisited"

while head $\geq$ tail

    $v \leftarrow queue[tail], tail \leftarrow tail + 1$

    for all neighbours u of v

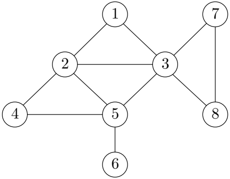
        if u is "unvisited" then

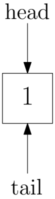
            $head \leftarrow head + 1, queue[head] = u$

            mark u as "visited"

- Running time: $O(n + m)$.

Example of BFS via Queue





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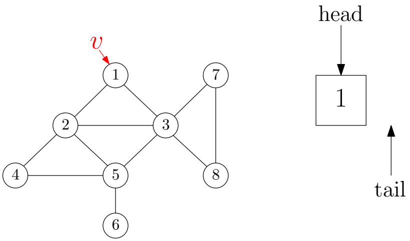

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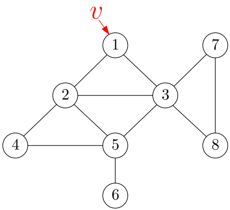

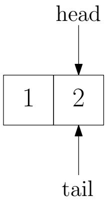

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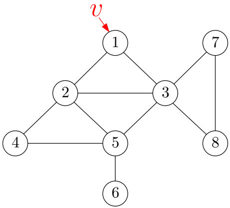

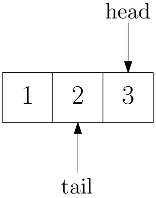

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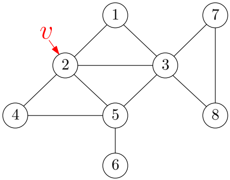

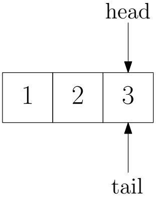

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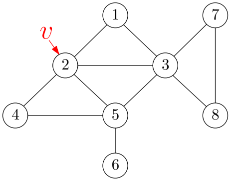

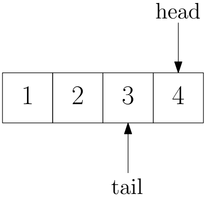

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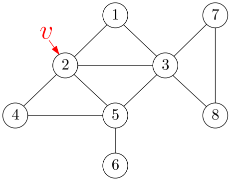

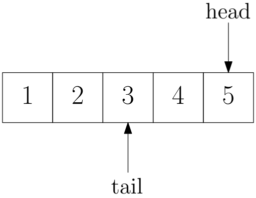

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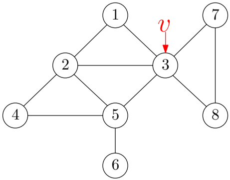

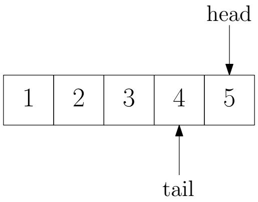

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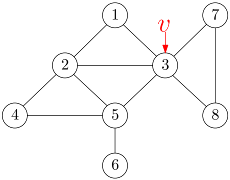

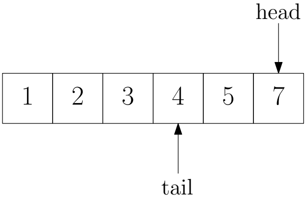

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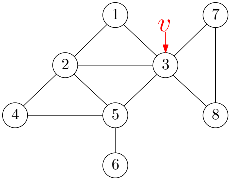

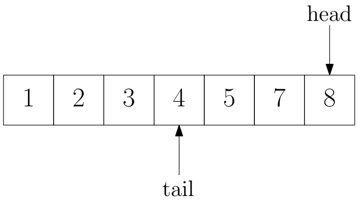

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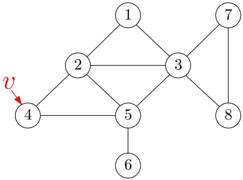

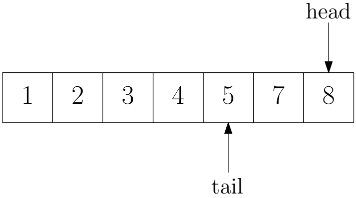

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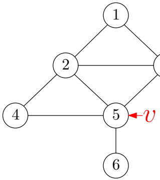

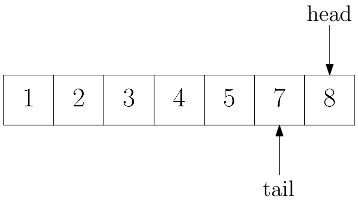

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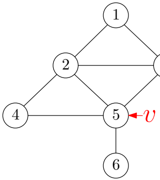

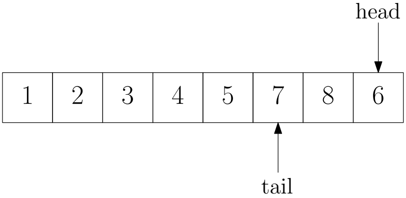

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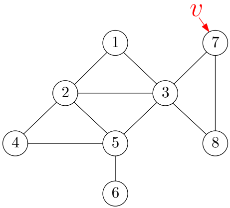

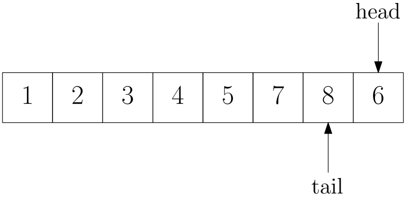

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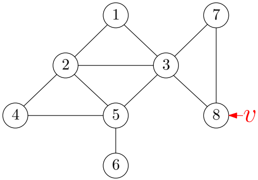

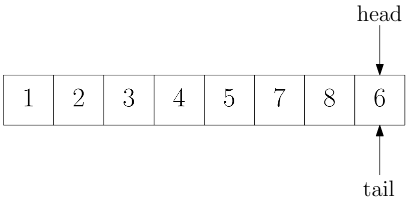

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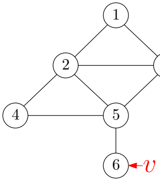

head

1	2	3	4	5	7	8	6

tail

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex ("dead-end"), go back
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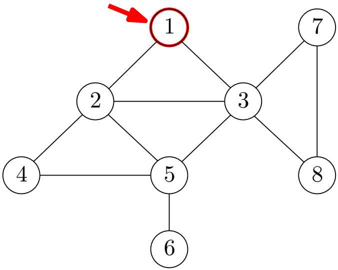

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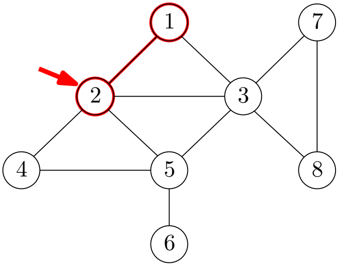

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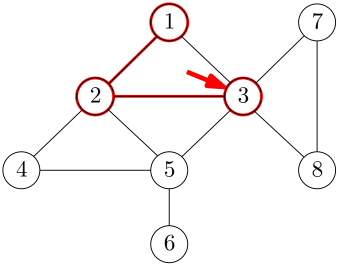

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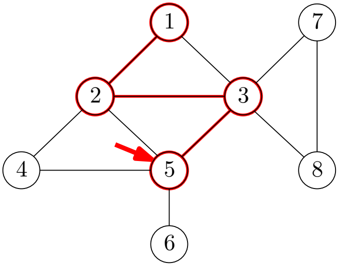

Depth-First Search (DFS)

- Starting from s
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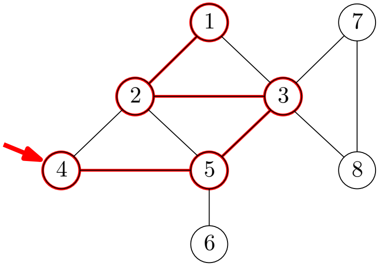

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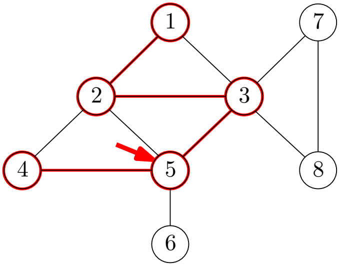

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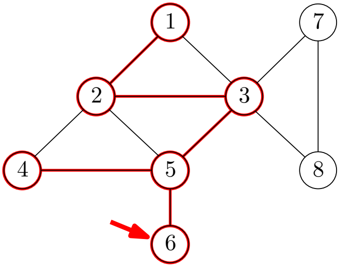

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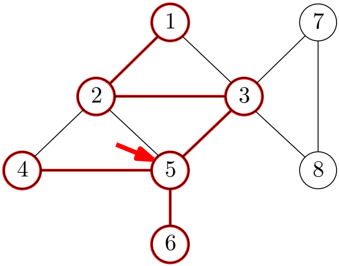

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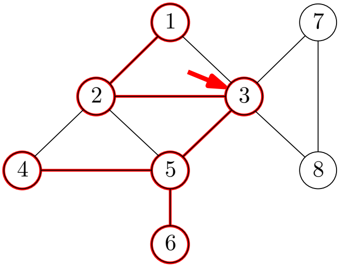

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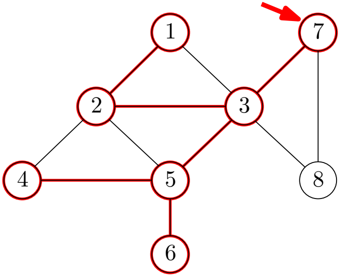

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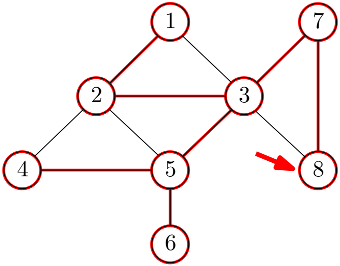


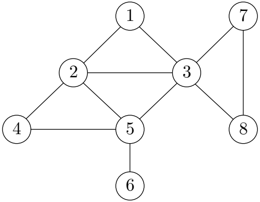
Implementing DFS using a Stack

DFS(s)

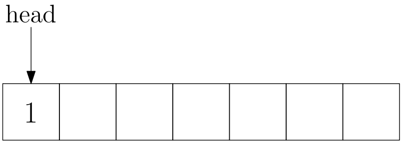
1. $head \leftarrow 1, stack[1] \leftarrow s$
2. mark all vertices as "unexplored"
3. while head $\geq$ 1
4. $\quad v \leftarrow stack[head], head \leftarrow head - 1$
5. $\quad$ if v is unexplored then
6. $\quad\quad$ mark v as "explored"
7. $\quad\quad$ for all neighbours u of v
8. $\quad\quad\quad$ if u is not explored then
9. $\quad\quad\quad\quad head \leftarrow head + 1, stack[head] = u$

- Running time: $O(n+m)$.

Example of DFS using Stack



explored vertices:



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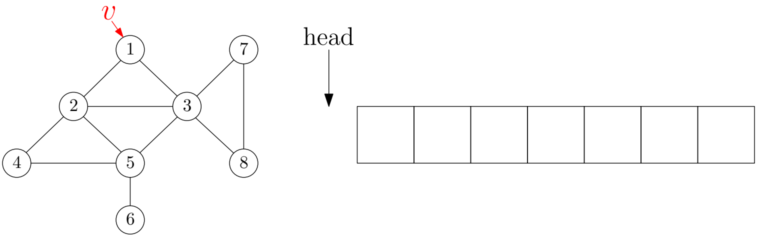
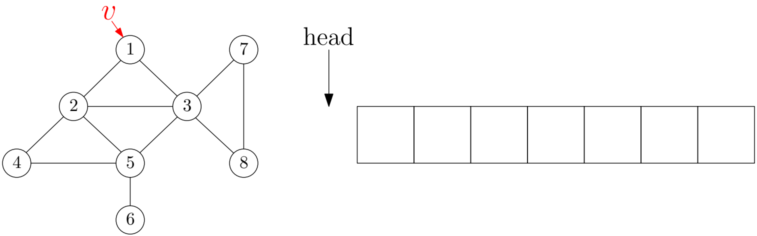


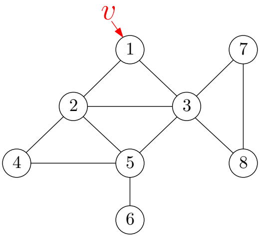
explored vertices:

Example of DFS using Stack

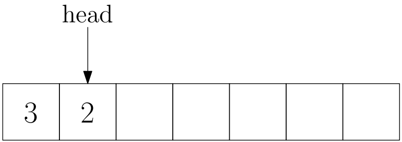


explored vertices: 1

Example of DFS using Stack



explored vertices: 1



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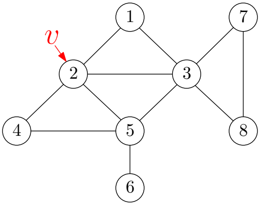
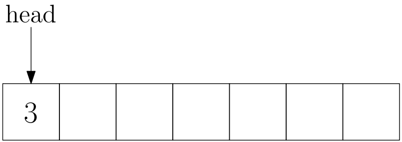


explored vertices: 1



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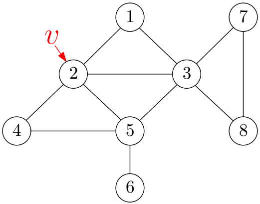
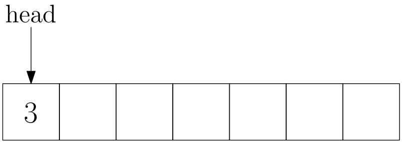


explored vertices: 1 2



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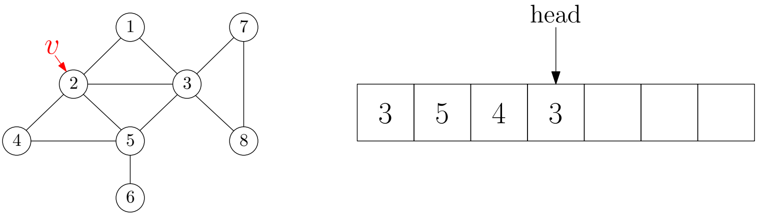

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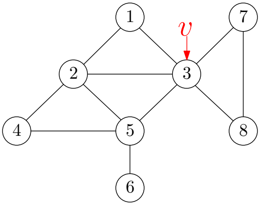

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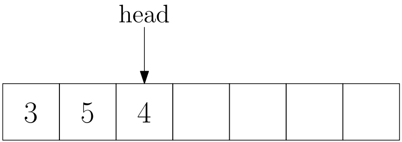

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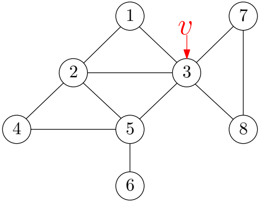

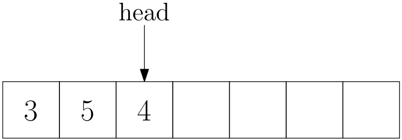

explored vertices: 1 2 3

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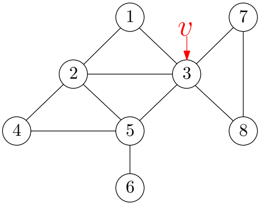
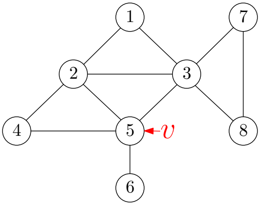


head

3	5	4	8	7	5	

explored vertices: 1 2 3

Example of DFS using Stack



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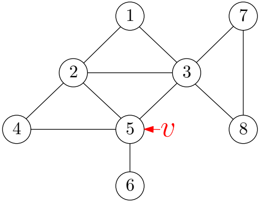
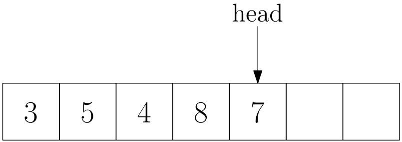
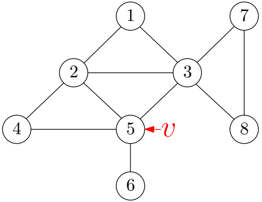
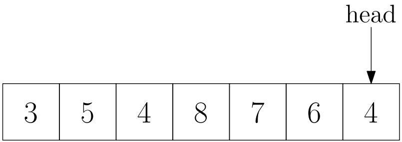




explored vertices: 1 2 3 5

Example of DFS using Stack





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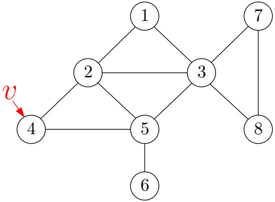

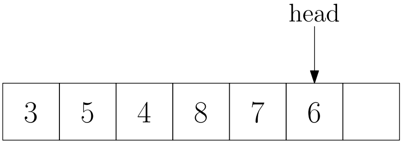

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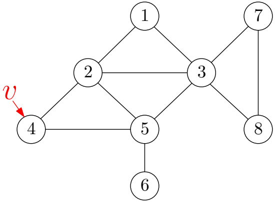

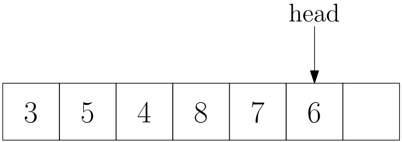

explored vertices: 1 2 3 5 4

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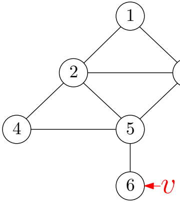

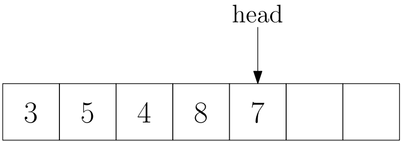

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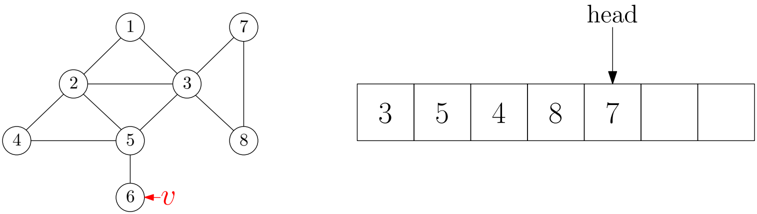

explored vertices: 1 2 3 5 4 6

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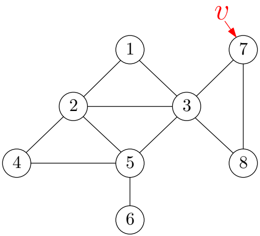

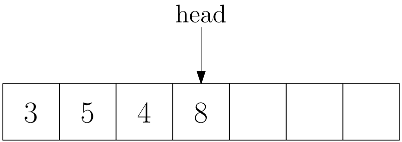

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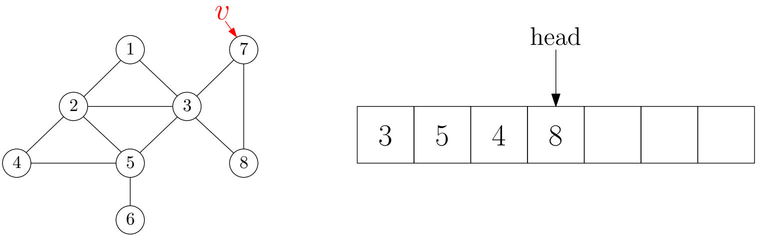

explored vertices: 1 2 3 5 4 6 7

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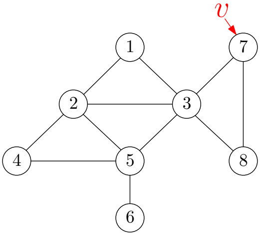

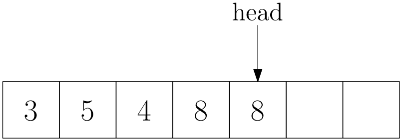

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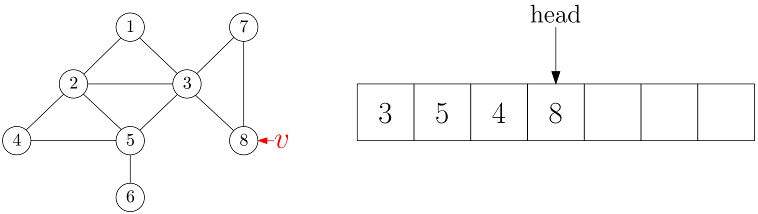

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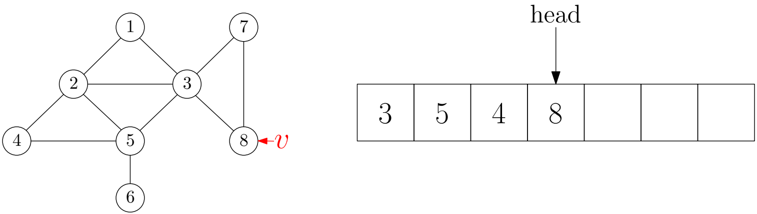

explored vertices: 1 2 3 5 4 6 7 8

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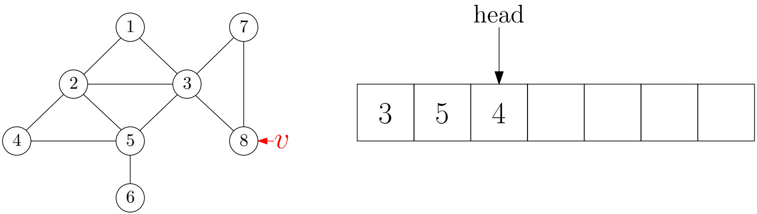

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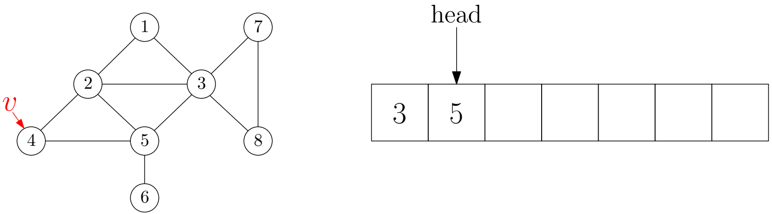

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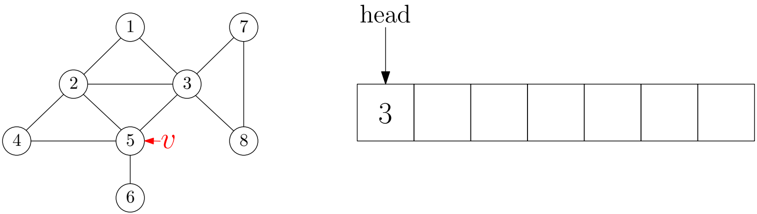

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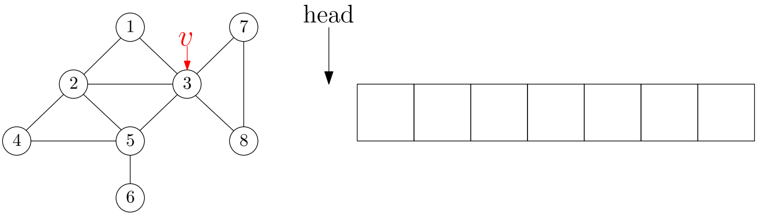


explored vertices: 1 2 3 5 4 6 7 8

Implementing DFS using Recurrsion

DFS(s)

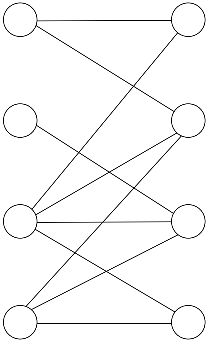
1. mark all vertices as "unexplored"
2. recursive-DFS(s)

recursive-DFS(v)

1. if v is explored then return
2. mark v as "explored"
3. for all neighbours u of v
4.   recursive-DFS(u)

Outline

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.



Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$

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- Assuming $s \in L$ w.l.o.g

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- Taking an arbitrary vertex $s \in V$
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- $\cdots$

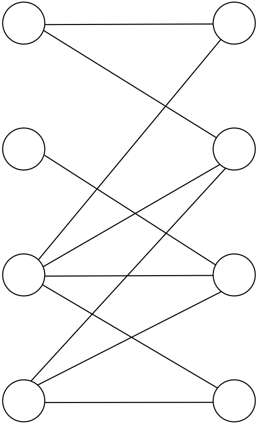
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- $\cdots$
- Report "not a bipartite graph" if contradiction was found

Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- $\cdots$
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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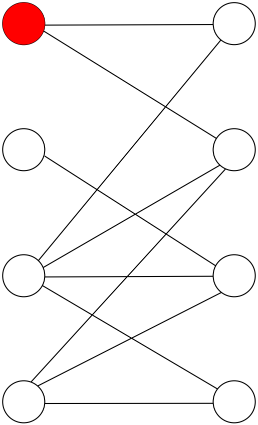

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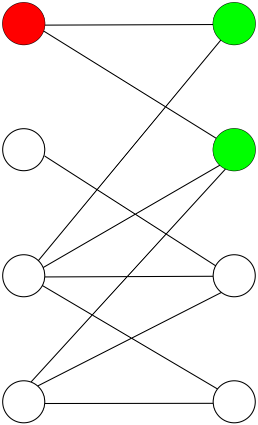

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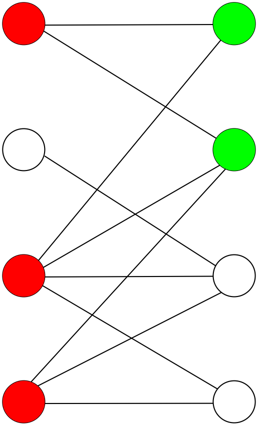

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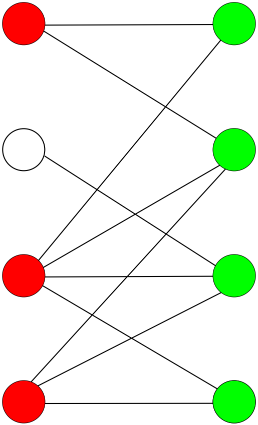

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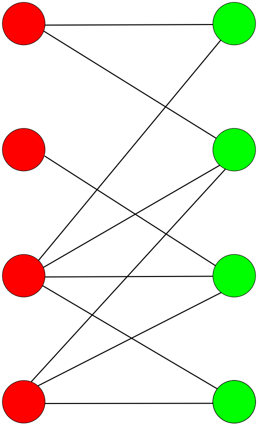

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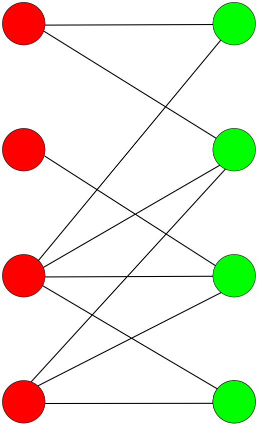

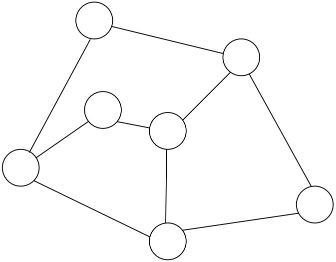

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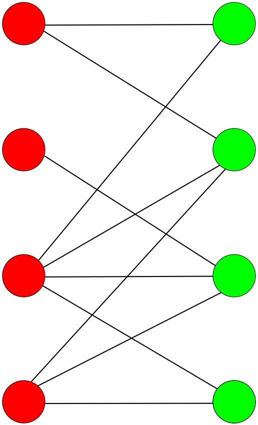

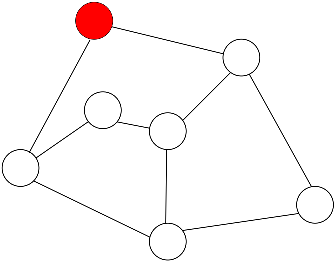

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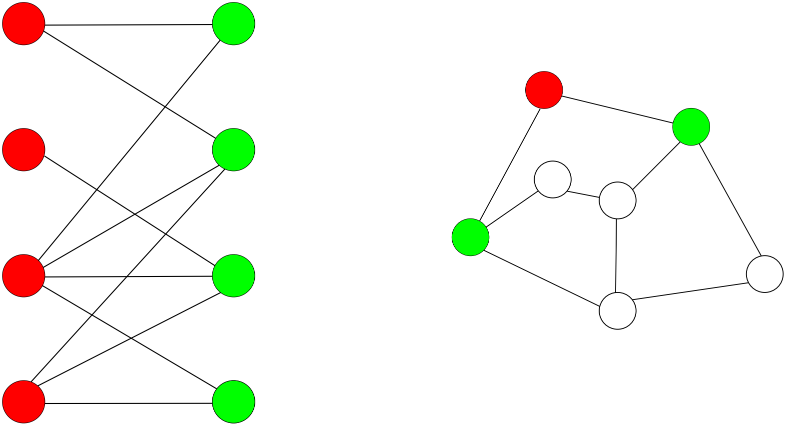

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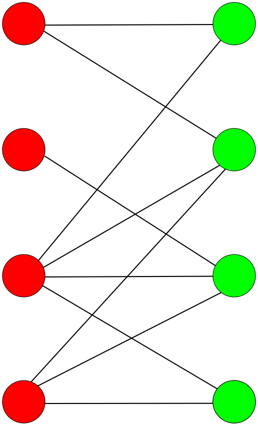
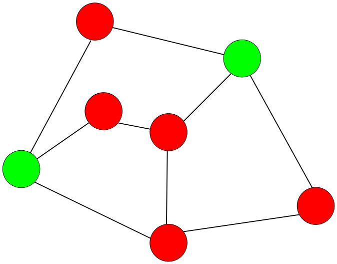
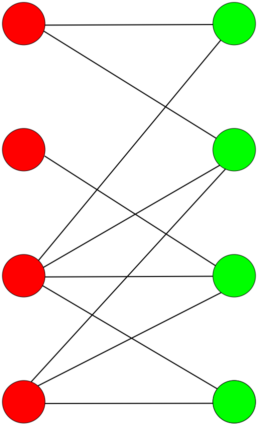
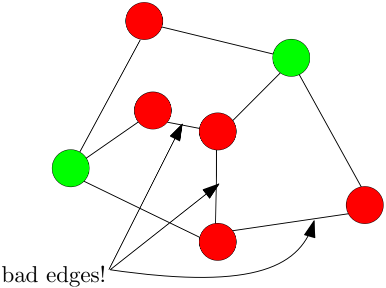




Test Bipartiteness





Testing Bipartiteness using BFS

BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark s as "visited" and all other vertices as "unvisited"
while head $\geq$ tail
    $v \leftarrow queue[tail], tail \leftarrow tail + 1$
    for all neighbours u of v
        if u is "unvisited" then
            $head \leftarrow head + 1, queue[head] = u$
            mark u as "visited"

Testing Bipartiteness using BFS

test-bipartiteness(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$

mark s as "visited" and all other vertices as "unvisited"

$color[s] \leftarrow 0$

while head $\geq$ tail

    $v \leftarrow queue[tail], tail \leftarrow tail + 1$

    for all neighbours u of v

        if u is "unvisited" then

            $head \leftarrow head + 1, queue[head] = u$

            mark u as "visited"

            $color[u] \leftarrow 1 - color[v]$

        elseif $color[u] = color[v]$ then

            print("G is not bipartite") and exit

Testing Bipartiteness using BFS

1. mark all vertices as "unvisited"
2. for each vertex $v \in V$
3.   if v is "unvisited" then
4.   test-bipartiteness(v)
5. print("G is bipartite")

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Homework problem: using DFS to implement test-bipartiteness.

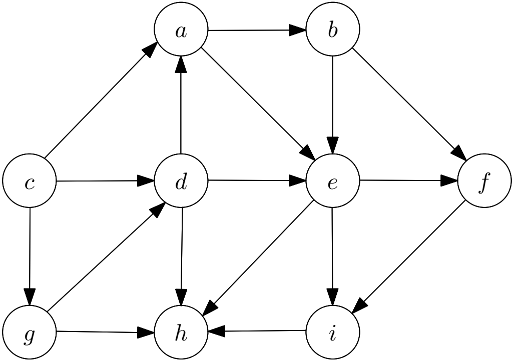
Outline

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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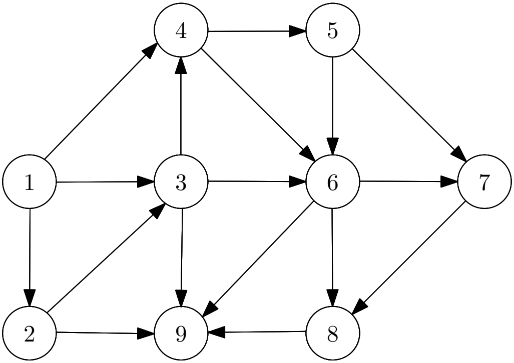
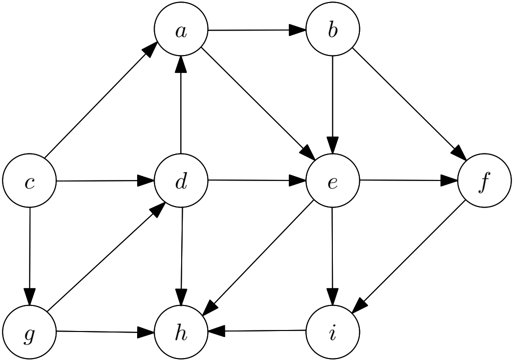


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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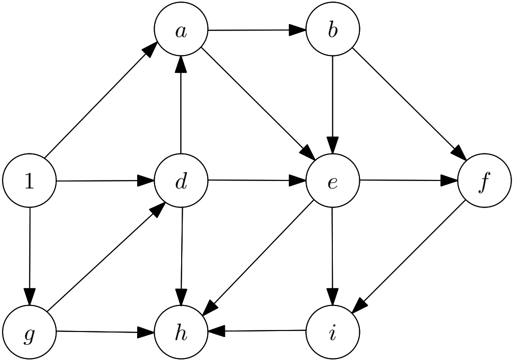

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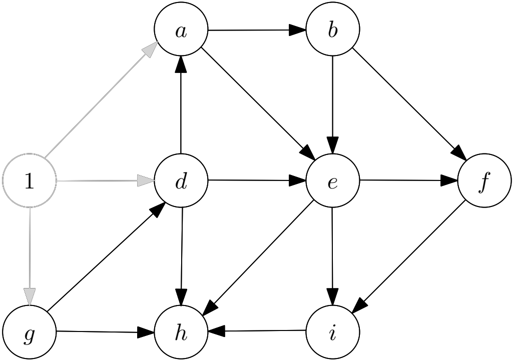

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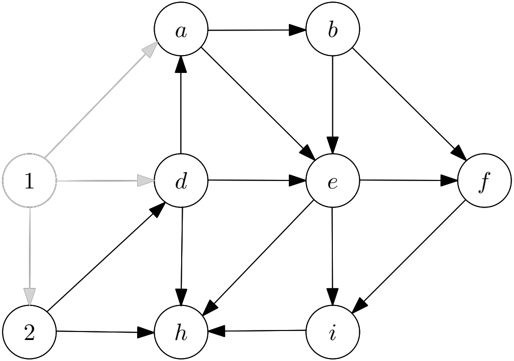

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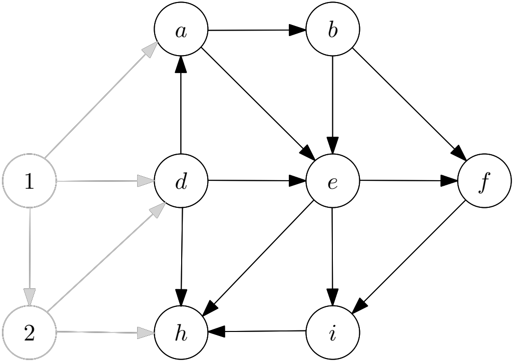

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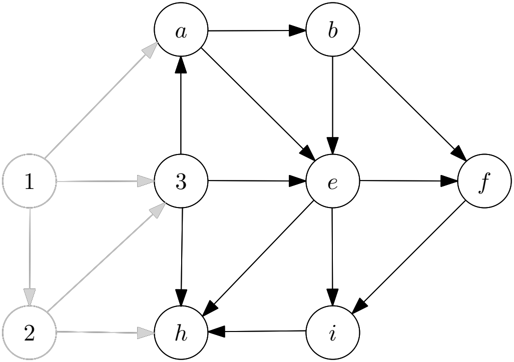

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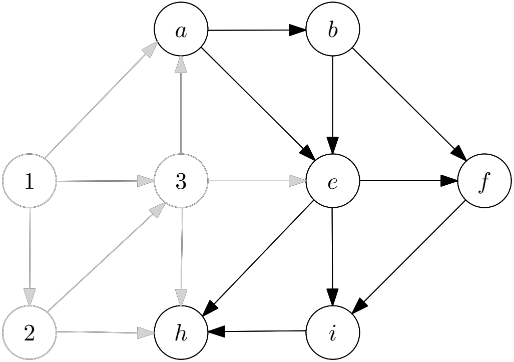

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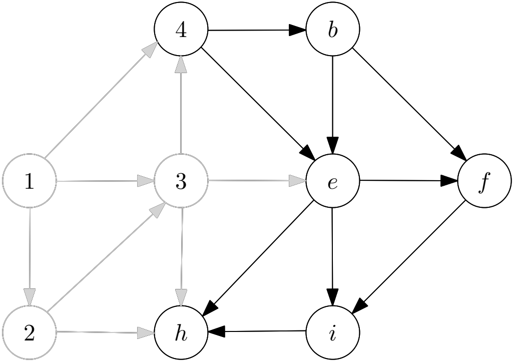

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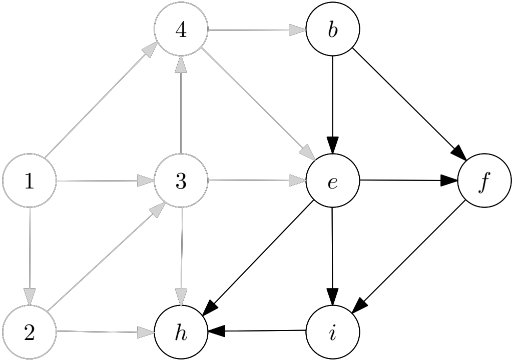

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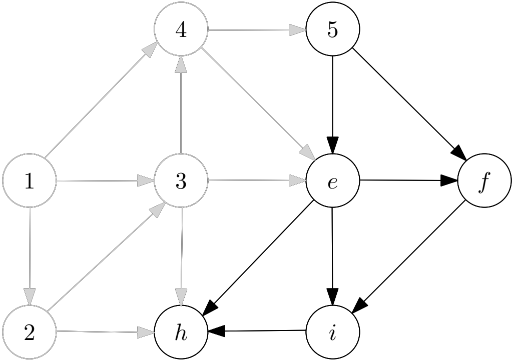

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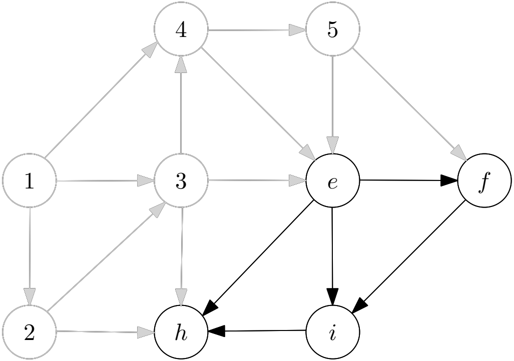

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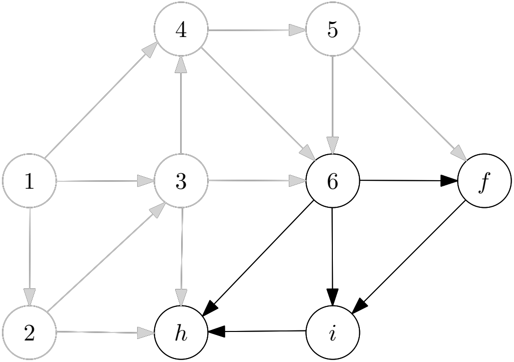

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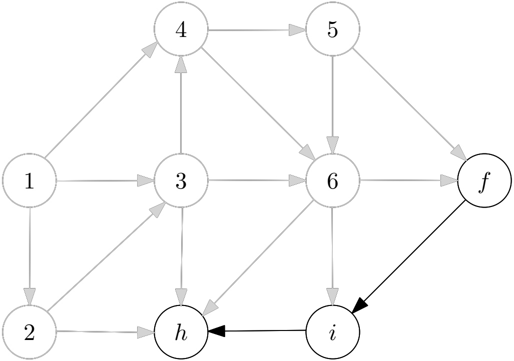

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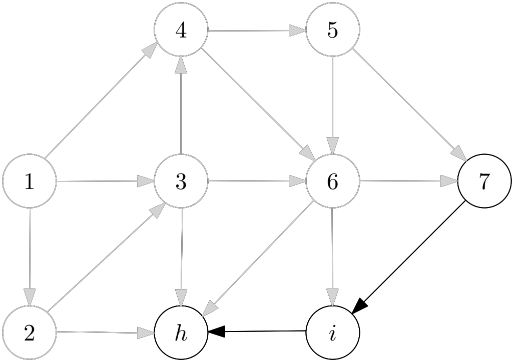

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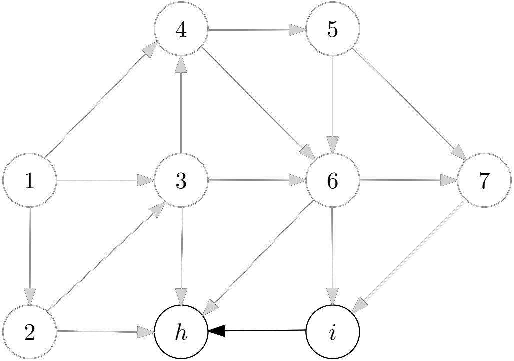

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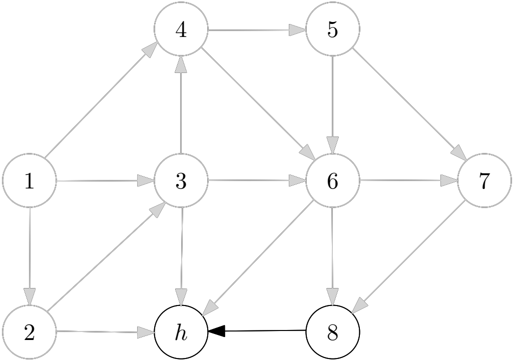

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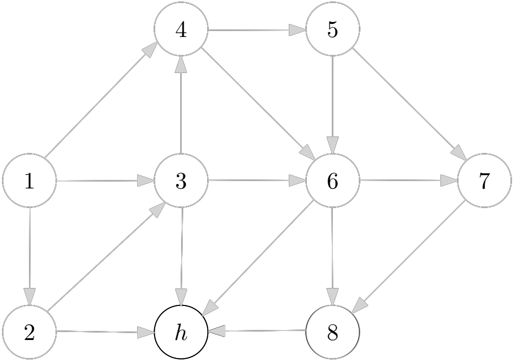

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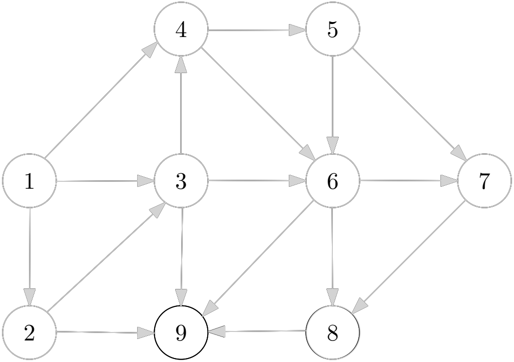

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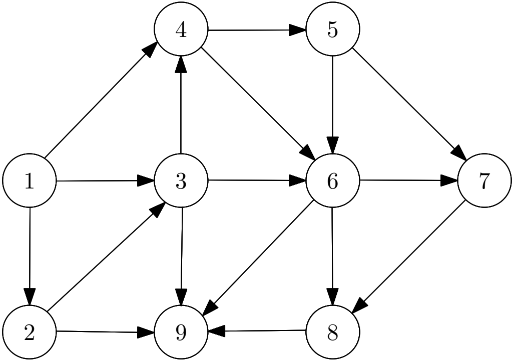

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Q: How to make the algorithm as efficient as possible?

Topological Ordering

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$
    for every u such that $(v, u) \in E$
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$
    $v \leftarrow$ arbitrary vertex in S, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every u such that $(v, u) \in E$
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ then add u to S
if $i < n$ then output "not a DAG"

- S can be represented using a queue or a stack
- Running time $= O(n + m)$